QCD at non-zero isospin asymmetry and its physical relevance

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in collaboration with
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DFG Deutsche Forschungsgemeinschaft

CRC-TR 211
Strong-interaction matter under extreme conditions

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1. Introduction:

(Lattice) QCD and isospin chemical potential

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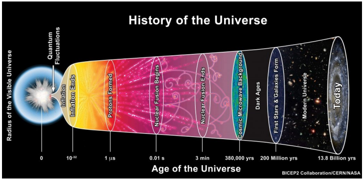
Motivation

theoretical first principles description of:

- evolution of early Universe

  (in particular: quark and hadron epochs)



[Credit: BICEP2 collaboration/CERN/NASA*]

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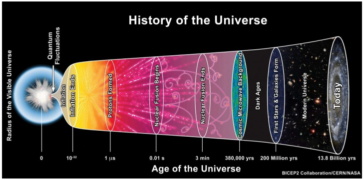

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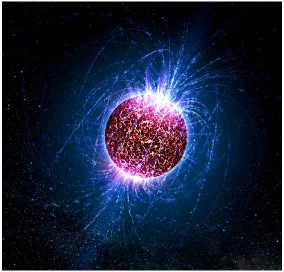

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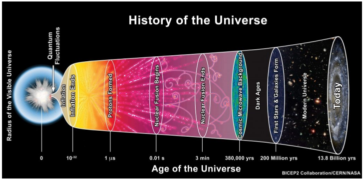

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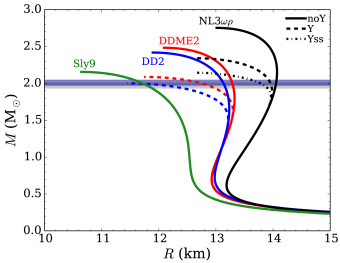
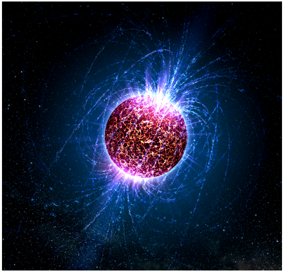


[Fortin, Providência *et al*, '16]



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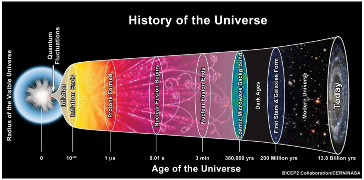
- fireball in heavy-ion collisions
  hydrodynamical treatment of equilibrated quark gluon plasma



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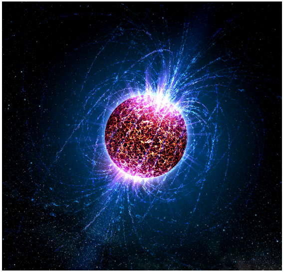

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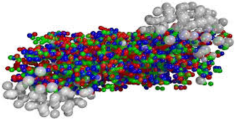

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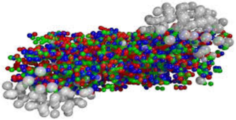
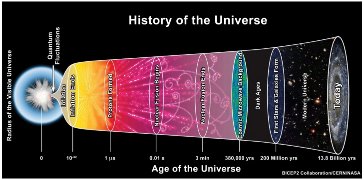
$\Rightarrow$ dominated by strong force

fundamental input:

- phase diagram
- equation of state (EoS)





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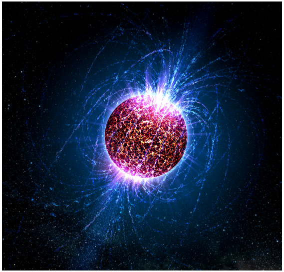

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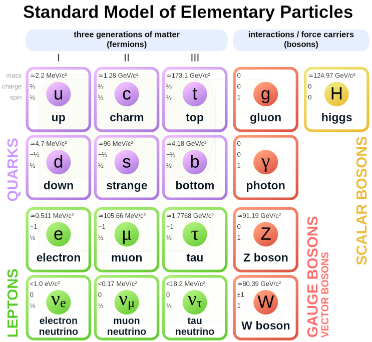
Introduction – Quantum Chromodynamics (QCD)

- theory of the strong interactions



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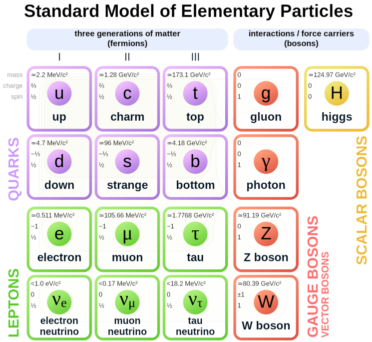
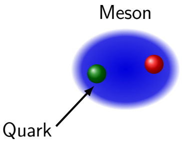
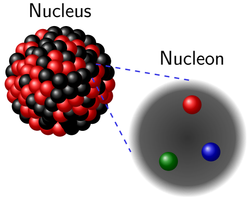
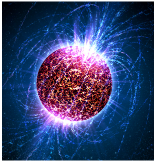
- theory of the strong interactions
- governs dynamics of smallest to largest boundstates

  (mesons & baryons to neutron star cores)







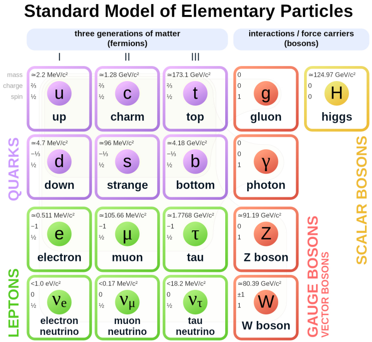


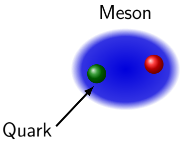
Introduction – Quantum Chromodynamics (QCD)

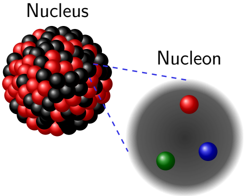
- theory of the strong interactions
- governs dynamics of smallest to largest boundstates

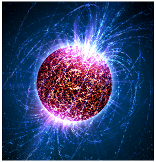
  (mesons & baryons to neutron star cores)
- responsible for 99.9% of the mass of visible matter in the Universe

	mass [MeV]		$\sum m_{\text{quark}}$ [MeV]
pion	≈ 135 MeV	$\longleftrightarrow$	≈ 7.2 MeV
nucleon	≈ 940 MeV	$\longleftrightarrow$	≈ 10.8 MeV









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Introduction – Quantum Chromodynamics

- QCD is a quantum field theory: particles $\longleftrightarrow$ field excitations
- QCD particles and fields:

quarks	u d s c t b	$\psi_f(x)$	$\in$ fundamental repr. of $SU(3)$
gluons	g	$A_\mu(x)$	$\in$ Lie algebra (su(3)) of $SU(3)$

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- QCD Lagrangian: (Euclidean spacetime – after Wick rotation)

$$\mathcal{L} = \underbrace{\sum_f \bar{\psi}_f \{\gamma_\mu (\partial_\mu + i g_s A_\mu) + m_f\} \psi_f}_{\text{fermionic}} + \underbrace{\frac{1}{4} \mathrm{Tr}\big(F_{\mu\nu} F_{\mu\nu}[A_\mu, g_s]\big)}_{\text{Yang-Mills (gluonic)}}$$

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- Euclidean path integral (PI) quantisation: [Feynman '48]

$$\mathcal{Z} = \int DAD\psi D\bar{\psi}\ \exp\Big(-\int d^4x\mathcal{L}(x)\Big)$$

- finite temperature partition function: (grand canonical – zero density)

$$\mathcal{Z} = \mathrm{Tr}\big[e^{-H/T}\big] = \int_{\mathrm{BC}} DAD\psi D\bar{\psi}\ \exp\Big(-\int_0^{1/T}\int d^3x\mathcal{L}(x)\Big)$$

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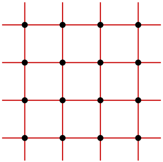
Introduction – lattice discretisation

- natural regulator for PI: discrete spacetime

[Wilson '74]



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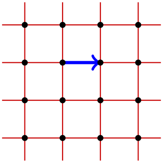

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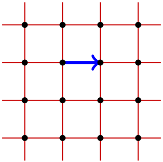

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- PI: fermions integrated out

$$\mathcal{Z} = \int [dU] \; \det\big(D[U]\big) \; e^{-S_{\mathrm{gluon}}[U]}$$

gluon action and Dirac operator (staggered):

$$S_{\mathrm{gluon}} = \frac{\beta}{N_c} \sum_p \mathrm{ReTr}\{\square\} \qquad D = \sum_{x,f} \left\{ m_f + \frac{\eta_\mu}{2a} \big(\longrightarrow - \longleftarrow \big)_\mu \right\}$$

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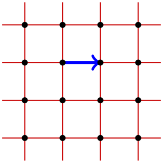

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- in finite volume: finite dimensional integral

  (use a $N_t \times N_s^3$ lattice – periodic spatial BC)

$\Rightarrow$ Monte-Carlo methods – importance sampling

$\mathcal{O}(10^9)$ degrees of freedom

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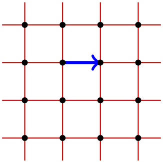

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  $\mathcal{O}(10^9)$ degrees of freedom $\rightarrow$ use supercomputer

[Copyright: Forschungszentrum Jülich]

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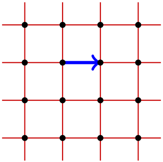

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[Copyright: NVIDIA (from nvidia.com)]

Introduction – Running coupling and Confinement

- QFT: physical quantities are renormalised quantities

  $\Rightarrow$ physical coupling $\alpha_s = g_s^R$ depends on the energy scale μ_E

  $\alpha_s \to 0$ for $\mu_E \to \infty$ asymptotic freedom [Gross, Wilczek '73; Politzer '73]

  $\alpha_s \to \infty$ for $\mu_E \to 0$

  typical energy scales: $g_s^R = \mathcal{O}(1)$

  $\Rightarrow$ non-perturbative methods are needed

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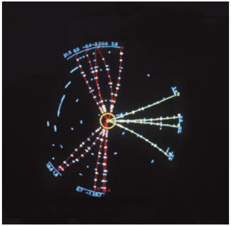
- Confinement
  - no direct evidence for QCD particles in colliders

    indirect evidence: jet events

    $\to$ quarks and gluons are confined in hadrons



[TASSO DESY (Image: Oxford PPU)]

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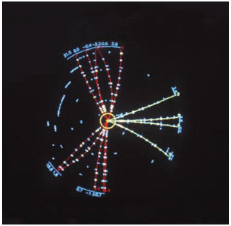

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- Confinement
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    indirect evidence: jet events

    $\rightarrow$ quarks and gluons are confined in hadrons
  - heuristic explanation for quark confinement:

    flux tube between quark and antiquark

  (evidence from pure gauge theory simulations [Lang, Rebbi '82; ...])

Introduction – chiral symmetry

- fundamental global symmetry for most of the phenomena observed in QCD:

chiral symmetry $\mathrm{SU}_V(2) \times \mathrm{U}_V(1) \times \mathrm{SU}_A(2) \times \mathrm{U}_A(1)$ $(N_f = 2)$

$\mathrm{SU}_V(2)$	$\mathrm{U}_V(1)$	$\mathrm{SU}_A(2)$	$\mathrm{U}_A(1)$
intact	intact Baryon number	broken spontaneously $\Rightarrow \langle\bar{\psi}\psi\rangle \neq 0$	broken anomalously

chiral limit $(m_{u/d} = 0)$

$\Rightarrow$ 3 Goldstone bosons – pions

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broken explicitly $m_u - m_d \neq 0$ — intact Baryon number — broken explicitly $m_{u/d} \neq 0 \quad \Rightarrow \quad \langle \bar{\psi}\psi \rangle \neq 0$

in nature (physical point)

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- low energy effective theory for Goldstone bosons:

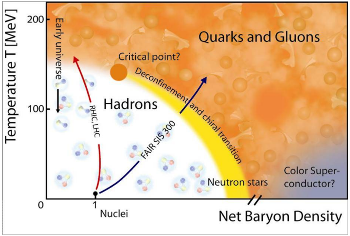
chiral perturbation theory (χPT)

  - systematic expansion in quark masses
    (external parameters (T, B, μ, ...) can be included)
  - valid as long as m_f, p, T, ... $< \Lambda_\chi$

Introduction – QCD phase diagram

At large T: quarks and gluons are liberated

$\Rightarrow$ plasma of quarks and gluons



[Homepage of the CMB experiment, GSI (FAIR)]

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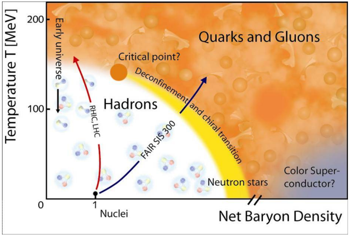
- transition at $n = 0$: (first principles LQCD)
  - transition is a crossover [Aoki *et al* '06]
  - $T_c \approx 157$ MeV ($\sim 10^{12}$ K)
    [Borsanyi *et al* '10; Bazavov *et al* '19]



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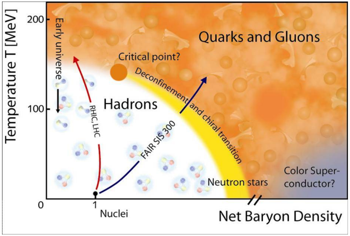
- deconfinement of quarks and gluons



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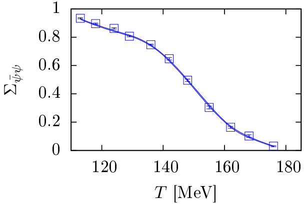

[data: Brandt, Endrődi '16]

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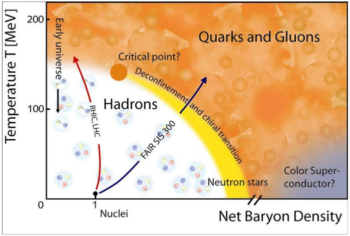
  driven by:
  - effective restoration of $SU_A(2)$
  - deconfinement of quarks and gluons
- transition at $n \neq 0$?

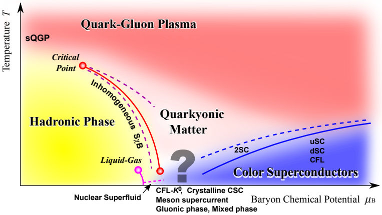
  models predict a number of phases

  possible 1st order phase trans.
  + critical endpoint(s)

  What about first principles information?



[Homepage of the CMB experiment, GSI (FAIR)]



[Fukushima, Hatsuda '11]

Isospin chemical potential

Lattice QCD: grand canonical ensemble density n $\rightarrow$ chemical potential μ

"Physical" basis for QCD at $N_f = 3$: (strangeness not conserved in SM)

$$\mu_u = \frac{\mu_B}{3} + \frac{2\mu_Q}{3} \qquad \mu_d = \frac{\mu_B}{3} - \frac{\mu_Q}{3} \qquad \mu_s = \frac{\mu_B}{3} - \frac{\mu_Q}{3} - \mu_S$$

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Convenient basis for simulations: ("isospin" basis)

$$\mu_u = \mu_L + \mu_I \qquad \mu_d = \mu_L - \mu_I \qquad \mu_s$$

relation between bases:

$$\mu_B = 3\mu_L - \mu_I \qquad \mu_Q = 2\mu_I \qquad \mu_S = \mu_L - \mu_I - \mu_s$$

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- $\mu_L \neq 0 \neq \mu_s$: complex action (sign) problem

$$\mathcal{Z} = \int [dU] \det \left(D[U](\mu_u, \mu_d, \mu_s)\right) e^{-S_{\mathrm{gluon}}[U]} \quad \in \mathbb{C}$$

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- pure isospin chemical potential: $\mu_L = \mu_s = 0$

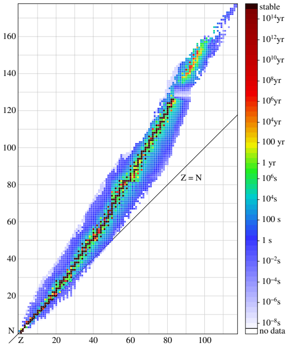
  $\det(D)$ is real, positive definite $\longrightarrow$ suitable for importance sampling

relation between bases:

$$\mu_B = 3\mu_L - \mu_I \qquad \mu_Q = 2\mu_I \qquad \mu_S = \mu_L - \mu_I - \mu_s$$

Isospin asymmetry – physical significance

isospin asymmetry: $n_I = n_u - n_d \neq 0$



- stable isotopes
- heavy-ion collisions (Pb or Au)

  $N_p/N_n \approx 2/3$
- neutron star cores [Steiner *et al* '05]

  $N_p/N_n \gtrsim 1/39$
- early universe?

  $n_I \neq 0$ possible for large lepton flavour asymmetries

  [Abuki, Brauner, Warringa '09; Wygas, Oldengott, Bödeker, Schwarz '18]

  $\Rightarrow$ discussed in more detail later

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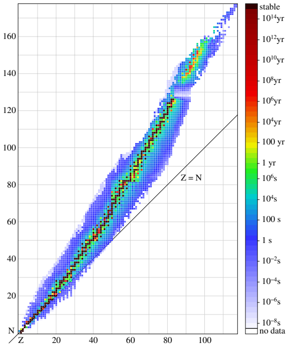


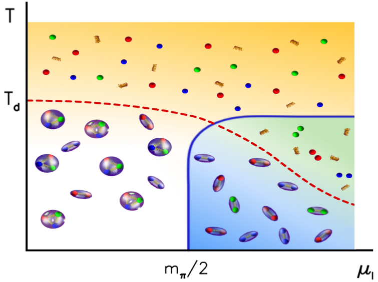
typically: n_B dominates (exception **eU**) however: n_I important ingredient

2. Phase diagram

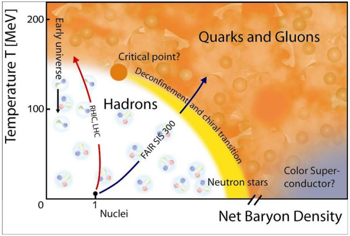
Phase diagram χPT [Son, Stephanov '01]

expected phase diagram:

- hadronic phase (white)
- quark-gluon plasma



[Brandt, Endrődi, Schmalzbauer '18]



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  condensation of charged pions
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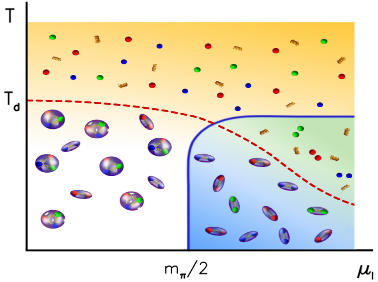

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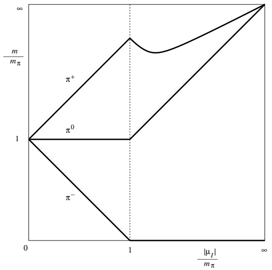

[Son, Stephanov '01]

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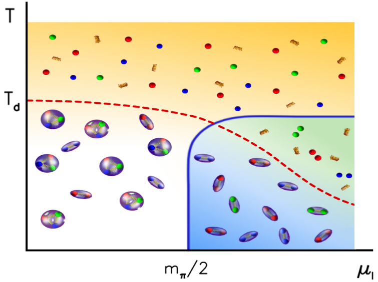
  condensation of charged pions
  (Bose-Einstein Condensation – BEC)
- $\mu_I \gg m_\pi/2$: BCS phase!?

perturbation theory:
1-gluon exchange is attractive



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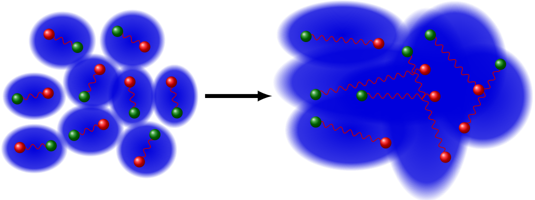

Chiral symmetry breaking pattern

$$D = \gamma_\mu D_\mu + m_{ud} + \gamma_0 \tau_3 \, \mu_I$$

$$\mathrm{SU}_V(2) \longrightarrow \mathrm{U}_Q(1)$$

explicit
$\mu_I \neq 0$

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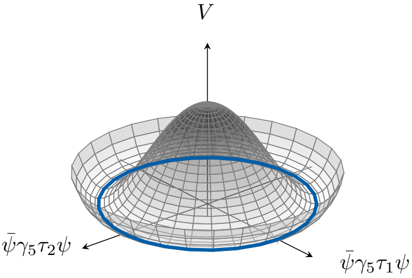

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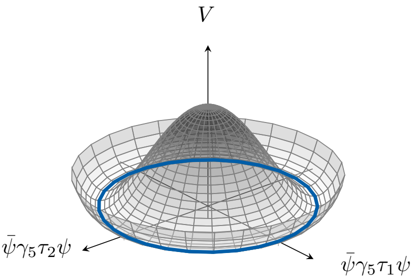
- cannot observe spontaneous symmetry breaking in finite V
- low mode in simulations



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Chiral symmetry breaking pattern

$$D = \gamma_\mu D_\mu + m_{ud} + \gamma_0 \tau_3 \, \mu_I + i \gamma_5 \tau_2 \, \lambda$$

$$\mathrm{SU}_V(2) \xrightarrow{} \mathrm{U}_Q(1) \xrightarrow{} \varnothing$$

explicit $\mu_I \neq 0$ — explicit pionic source λ

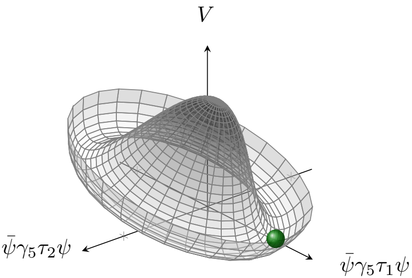
- cannot observe spontaneous symmetry breaking in finite V
- low mode in simulations

- need to break symmetry explicitly

  $\Rightarrow$ introduce regulator: $\sim \lambda$
  pionic source [Kogut, Sinclair '02]

  physical results: extrapolate $\lambda \to 0$

  reliable extrapolations:
  main task for analysis



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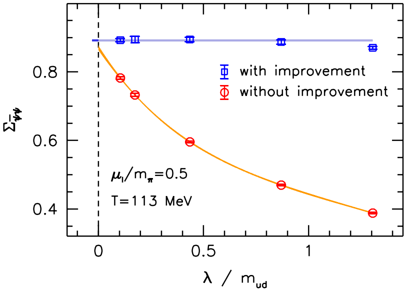
  physical results: extrapolate $\lambda \to 0$

  reliable extrapolations: main task for analysis
- improvement program:

  [Brandt, Endrődi, Schmalzbauer '18]

  - valence quark improvement
  - leading order reweighting



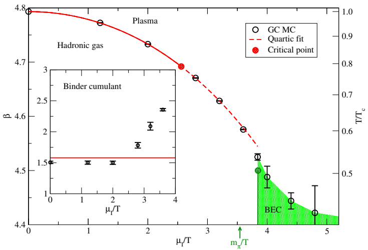
Status and setup

- model/EFT results: (recent review: [Mannarelli '19])
  - χPT
  - linear sigma model
  - (P)NJL phase diagram in hep-ph/0508117
  - HRG

  (for detailled references see [Brandt, Endrődi, Schmalzbauer '18])
- first results from lattice QCD:

  $N_t = 4$, unphysical masses, unimproved:
  - $N_f = 2$ [Kogut, Sinclair, '02; '04]
  - $N_f = 8$ [de Forcrand, et al, '07]

  canonical approach, unph. masses, $T = 0$:
  - $N_f = 2 + 1$ [Detmold, Orginas, Shi '12]

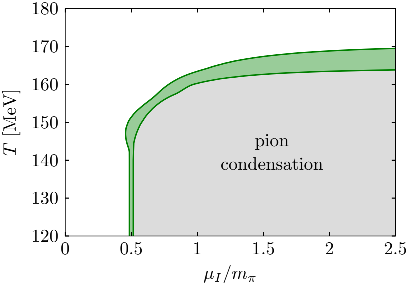


$N_f = 8$ [de Forcrand, et al, '07]

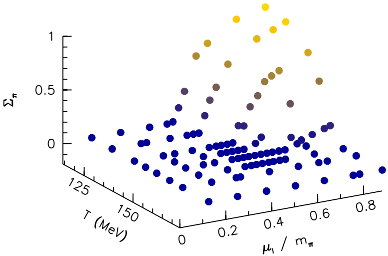
- here:
  - improved actions ($N_f = 2 + 1$)
  - physical quark masses
  - well controlled λ-extrapolations (from now on everything $\lambda = 0$)

Phase diagram from the lattice

- continuum phase diagram
  [Brandt, Endrődi, Schmalzbauer '18]
  - BEC phase boundary



- order parameter:
  renormalised pion condensate



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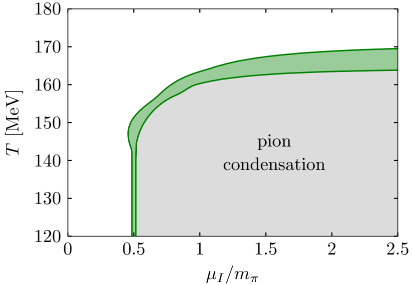
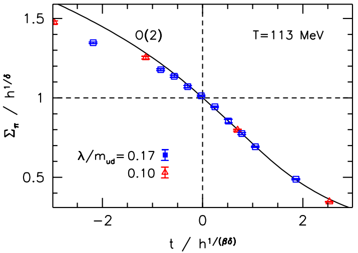
    $O(2)$ universality class



- order parameter:
  renormalised pion condensate
- finite size scaling:
  consistency with $O(2)$ scaling



Phase diagram from the lattice

- continuum phase diagram
  [Brandt, Endrődi, Schmalzbauer '18]
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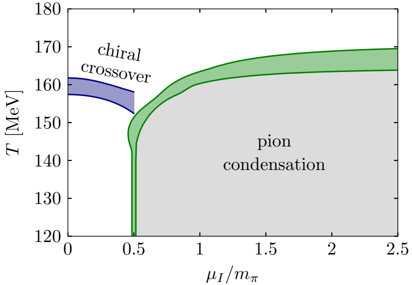
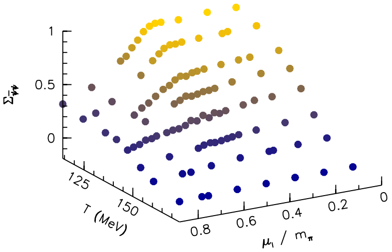
    $O(2)$ universality class
  - chiral crossover



- relevant observable:
  renormalised chiral condensate



Phase diagram from the lattice

- continuum phase diagram
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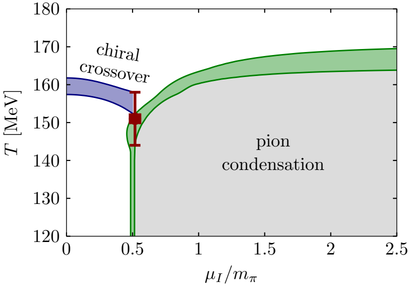
  - BEC phase boundary
    $O(2)$ universality class
  - chiral crossover
  - *pseudo-triple* point

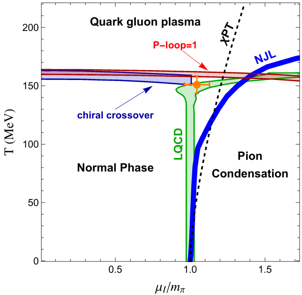


- meeting point:
  crossover and BEC phase boundary:

  pseudo coexistence of three phases
- from this point on:

  phase boundaries coincide

Comparison to model and EFT results



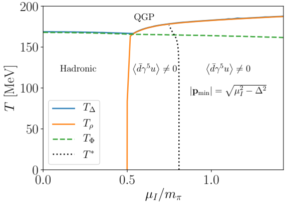
[Mannarelli '19]

χPT [Splittorff, Toublan, Verbaarschot '02]

NJL [He, Jin, Zhuang '05]

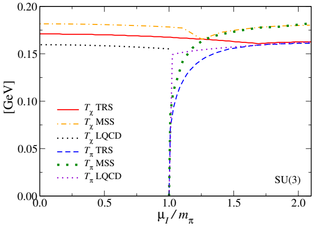
lattice results as model input at $\mu \neq 0$:

- calibrate models
- improve understanding of model capabilities



Polyakov quark meson model

[Folkestad, Andersen '19]



SU(3) NJL [Lopez *et al* '21]

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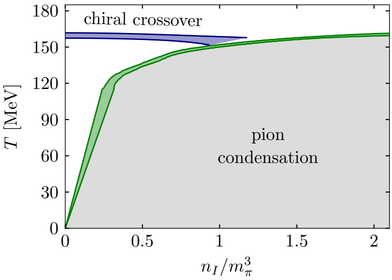
Phase diagram in the $n_I - T$ plane

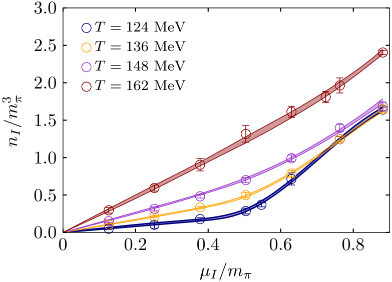
using a model independent 2d interpolation of n_I

(model independent: all possible “good” spline fits via Monte-Carlo)

[S. Borsanyi, private comm.; Brandt, Endrődi '16]

determine the $n_I - T$ phase diagram





$N_t = 8$

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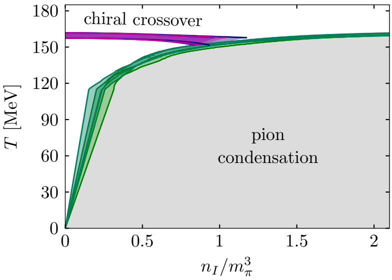

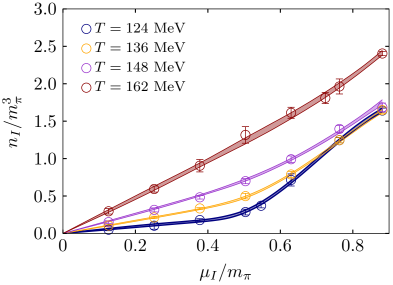

$N_t = 8$

- consider multiple N_t for continuum limit

  ... work in progress

3. Equation of state

Equation of state at $T = 0$

Grand canonical ensemble $T = 0$:

$$p(T = 0, \mu_I = 0) = 0 \qquad n_I = \frac{\partial p}{\partial \mu_I} \qquad \epsilon = -p + n_I \mu_I$$

$$\Rightarrow \quad p(0, \mu_I) = \int_0^{\mu_I} d\mu \, n_I(0, \mu_I) \qquad I = -4p + n_I(0, \mu_I)\mu_I$$

can be obtained from a spline interpolation of $n_I(0, \mu_I)$

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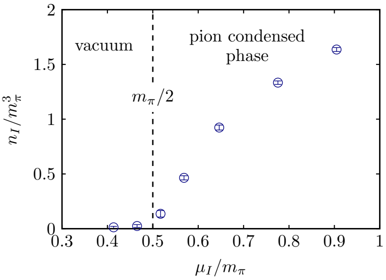
  $T = 0$ never exactly fulfilled

  $\Rightarrow$ use large N_t so that $T \approx 0$



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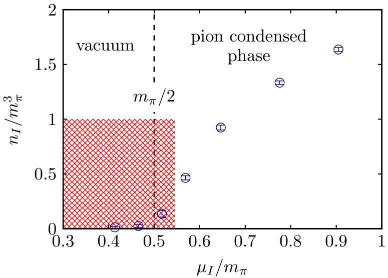
- correct for $T \neq 0$ effects with χPT

  particularly relevant for $\mu_I \approx m_\pi/2$



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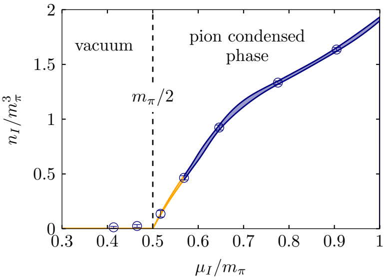
  particularly relevant for $\mu_I \approx m_\pi/2$

  $\Rightarrow$ fit at LO gives $f_\pi = 133(4)$ MeV



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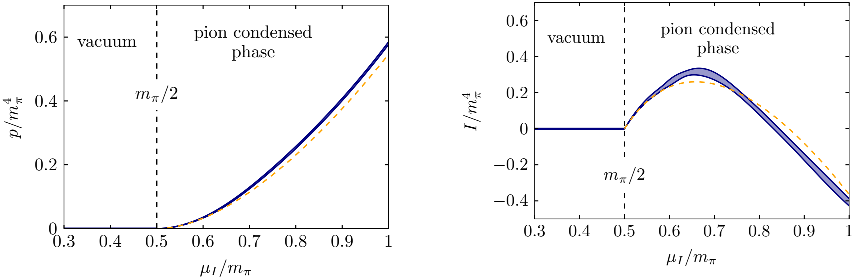
Equation of state at $T = 0$

Resulting equation of state:



- in good agreement with NLO χPT [Adhikari, Andersen '19]
- currently: only a single lattice spacing

continuum limit: . . . work in progress

Extracting the EoS at non-zero T

$p(T,0)$ and $I(T,0)$ are known [Borsanyi *et al* '13, Bazavov *et al* '14]

$\Rightarrow$ need to determine:

$$\Delta p(T,\mu_I) \equiv p(T,\mu_I) - p(T,0) \quad \text{and} \quad \Delta I(T,\mu_I) \equiv I(T,\mu_I) - I(T,0)$$

Extracting the EoS at non-zero T

$p(T,0)$ and $I(T,0)$ are known [Borsanyi *et al* '13, Bazavov *et al* '14]

$\Rightarrow$ need to determine:

$$\Delta p(T,\mu_I) \equiv p(T,\mu_I) - p(T,0) \quad \text{and} \quad \Delta I(T,\mu_I) \equiv I(T,\mu_I) - I(T,0)$$

- computation of Δp:

  As before: $$\Delta p(T,\mu_I) = \int_0^{\mu_I} d\mu\, n_I(T,\mu)$$

- computation of ΔI:

  Starting point: $$\frac{\Delta I(T,\mu_I)}{T^4} = T\frac{\partial}{\partial T}\left(\frac{\Delta p(T,\mu_I)}{T^4}\right) + \frac{\mu_I n_I(T,\mu_I)}{T^4}$$

  $\Rightarrow$
  $$\Delta I(T,\mu_I) = -4\int_0^{\mu_I} d\mu_I' n_I(T,\mu_I') + \int_0^{\mu_I} d\mu_I'\, T\frac{\partial}{\partial T} n_I(T,\mu_I') + \mu_I n_I(T,\mu_I)$$

use 2d interpolation of $n_I(T,\mu_I)$

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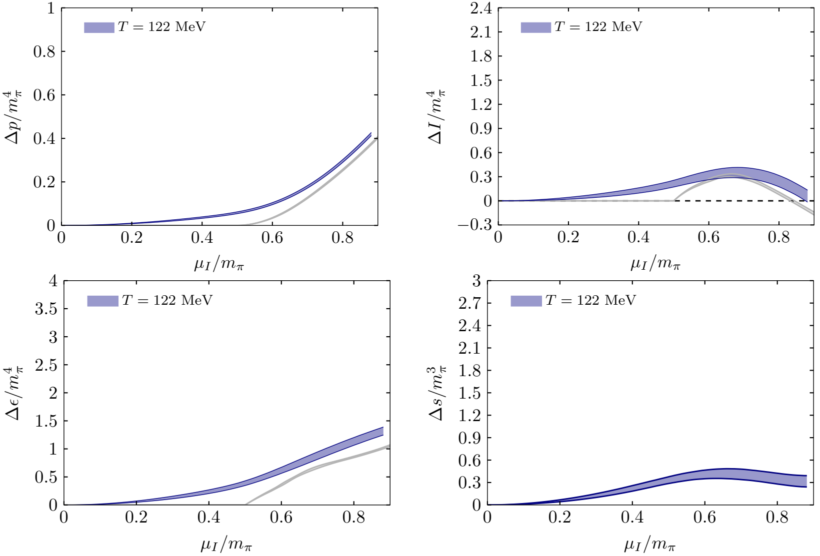
EoS $T \neq 0$ and $\mu_I \neq 0$: $N_t = 8$



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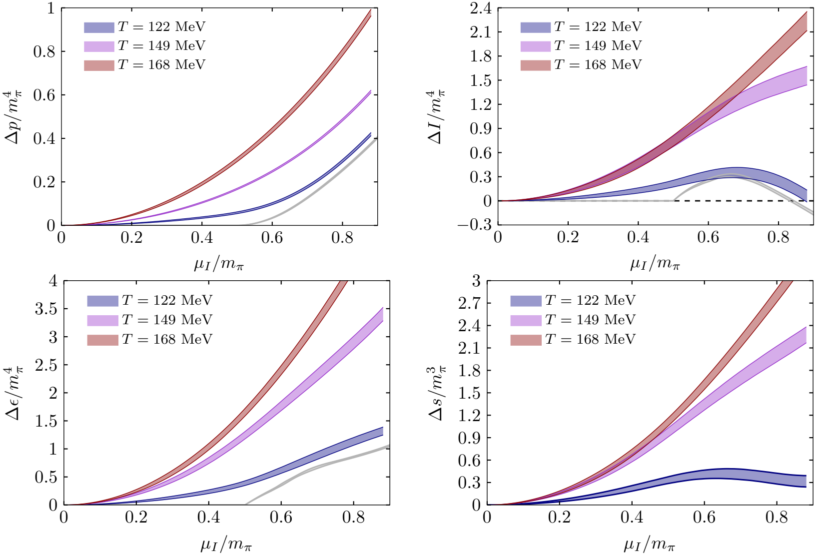
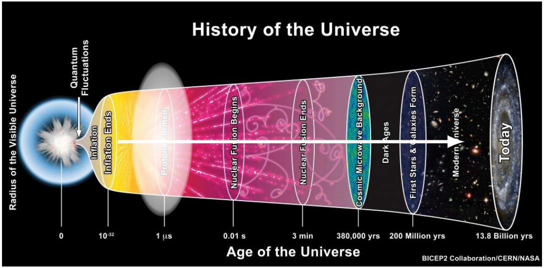
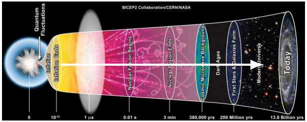


4. Early Universe at large lepton flavour asymmetries



Evolution of the early Universe



for $100\,\mathrm{GeV} > T \gtrsim 10\,\mathrm{MeV}$:

$n_B,\ n_Q,\ n_{L_e},\ n_{L_\mu},\ n_{L_\tau}$

conserved in comoving volume

- Cosmic trajectory: isentropic expansion with parameters
  $T,\ \mu_B,\ \mu_Q,\ \mu_{L_\ell}$ with $\ell \in (e, \mu, \tau)$
  (grand canonical ensemble)
- conservation equations:
  $$\frac{n_B}{s} = b \qquad \frac{n_Q}{s} = 0 \qquad \frac{n_{L_\ell}}{s} = l_\ell$$

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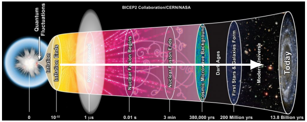

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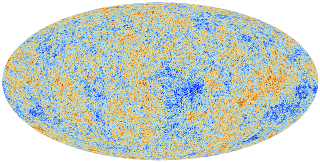
- empirical constraints (CMB):
  - $b = 8.6(0.06) \cdot 10^{-11}$ [Planck collaboration '15]
  - $|l_e + l_\mu + l_\tau| < 0.012$ [Oldengott, Schwarz '17]



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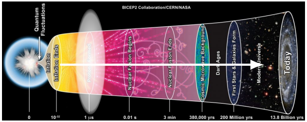

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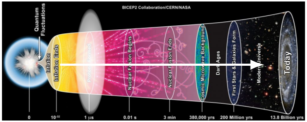

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EoS for cosmic trajectory:

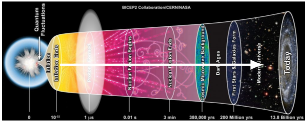
$$p \approx p_{\mathrm{QCD}} + p_{\mathrm{lept.}} + p_\gamma$$

ideal gas ℓ and ν_ℓ — ideal gas

p_{QCD} from the lattice

- to first approximation: EoS for $\mu_B = \mu_Q = 0$
- effects of $\mu_B \neq 0 \neq \mu_Q$: Taylor expansion

Cosmic trajectory with lepton flavour asymmetries



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▶ sum of l_ℓ constrained – individual l_ℓ not

$l_\ell - l_{\ell'} \gtrsim 0.1$ possible in models:

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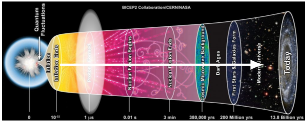

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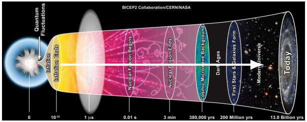
- cosmic trajectories for $|l_\ell| \gg 0$ (but $|l_e + l_\mu + l_\tau| = 0$)

  $\Rightarrow$ trajectories at large $\mu_Q \gg |\mu_B| > 0$

  - EoS from Taylor expansion

    [Middeldorf-Wygas, Oldengott, Bödeker, Schwarz '20]
  - EoS from model [Vovchenko, Brandt *et al* '20]

Cosmic trajectories and pion condensation



First test: simple model with pion condensation

- HRG & effective mass model for pions (quasiparticle picture & rearrangement term)
- match the model to lattice QCD at $T = 0$

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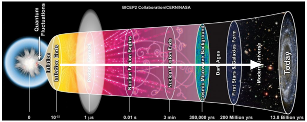

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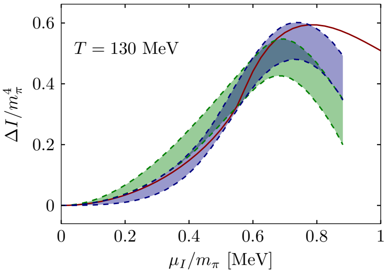
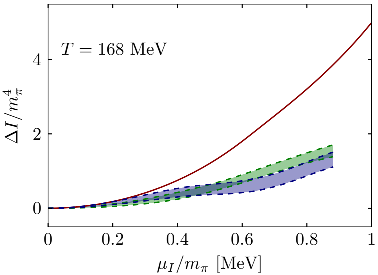
► expect the model to work reliably for: $T \lesssim 160$ MeV $\mu_I \lesssim 0.8 - 0.9\, m_\pi$





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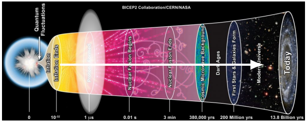

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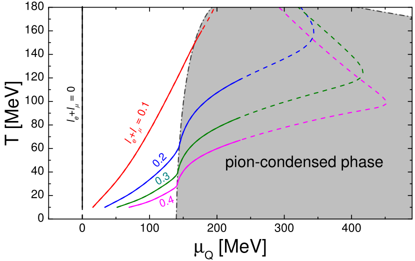
Cosmic trajectories with large $|l_e + l_\mu|$

constraints:

- $|l_e + l_\mu + l_\tau| = 0$
- $l_e - l_\mu = 0$
- dependence on $l_e - l_\mu$ very mild



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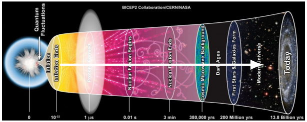

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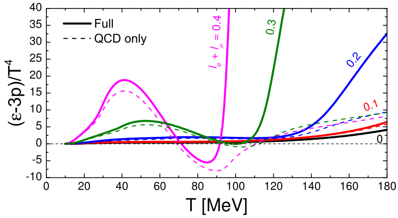
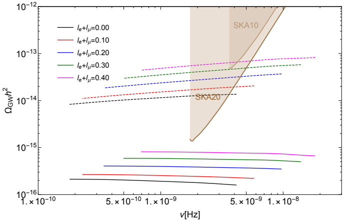
Effects of pion condensation:

- pion condensation strongly affects EoS
- enhances relic density of primordial gravitational waves
- modifies fraction of primordial black holes heavier than solar masses



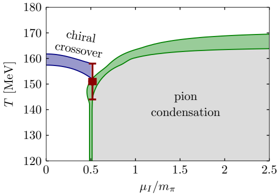


Conclusions

- QCD phase diagram at $\mu_I \neq 0$

  open question:

  existence/location of BCS phase?



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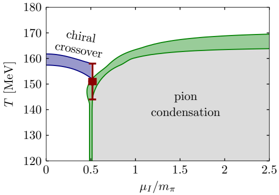
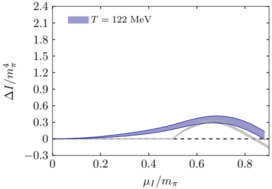
  work in progress: continuum limit





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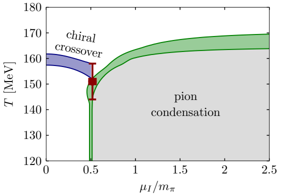

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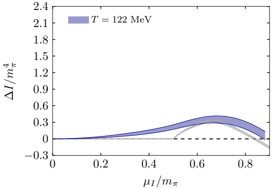


- early universe with lepton flavour asymmetry

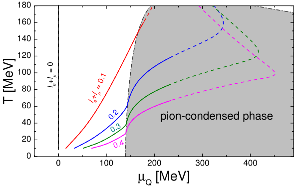
  [Vovchenko, Schaffner-Bielich, Hajkarim, Brandt *et al* '20]

  $\Rightarrow$ conceivable lepton flavour asymmetries:

  pion condensation can occur

  possible environment for pion stars?

  [Brandt, Endrődi, Fraga, Hippert *et al* '18]

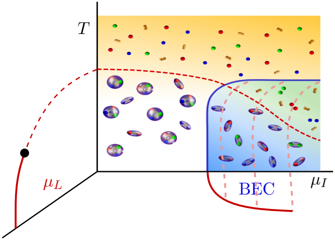


Outlook: extension to non-zero baryon density

Physical systems discussed:

all feature $\mu_L \neq 0 \neq \mu_s$

- early universe with large lepton asymm.
- compact stars
- heavy ion-collisions



eventually: need to overcome sign problem

for small $\mu_L \neq 0 \neq \mu_s$:

indirect methods: (starting from $\mu = 0$)
Taylor expansion; analytic continuation; reweighting
(only possible in absence of phase transition)

here: use results at $\mu_I \neq 0$ as a novel starting point for Taylor expansion

$\Rightarrow$ Work in progress ...

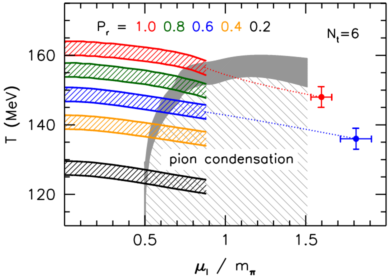
Hunting the BCS phase

- First evidence:

  coexistence of BEC and deconfinement

  $\Rightarrow$ look at Polyakov loop (associated with deconfinement)

  However: need a more clear-cut criterion



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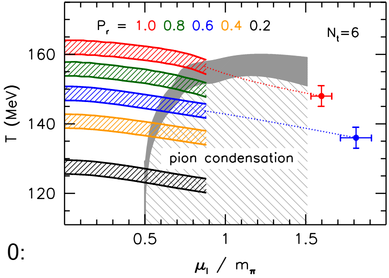

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$$\Delta^2 = \frac{2\pi^3}{9} \langle \rho(0) \rangle \qquad (V \to \infty,\ m_{ud} \to 0,\ \mu_I \text{ large})$$

[Kanazawa, Wettig, Yamamoto '12]

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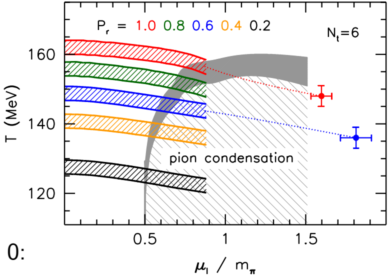

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  for $m_{ud} \neq 0$: $\quad \rho(0) \to \rho(m_{ud})$

  search plateau for $\rho(m_{ud})$ at large μ_I

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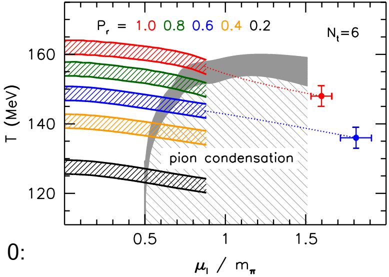

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Need:
- finer lattices
- larger μ_I
- $\lambda \to 0$ extrap.
- finite V study