Solving Quadratic Optimization Problems with Quantum Annealing

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Introduction

Quantum annealing and quadratic optimization

Quadratic Optimization Problems

+ A **Q**uadratic **U**nconstrained **B**inary **O**ptimization problem asks for

$$\underset{a_i,\, b_{ij}}{\arg\min} \left(\sum_i a_i \boldsymbol{q_i} + \sum_i b_{ij} \boldsymbol{q_i} \boldsymbol{q_j} \right)$$

where $q_i = 0, 1$.

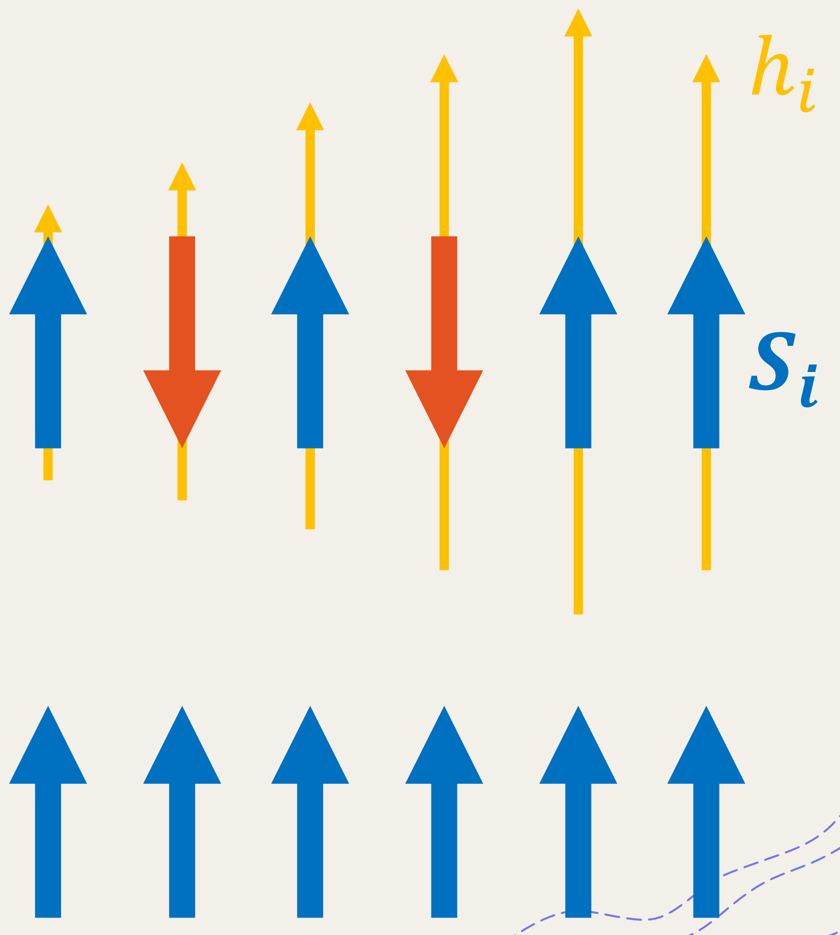
Ising Model

+ A chain of spins, $S_i = \pm 1$, s.t.

$$E = -\sum_i h_i S_i + \sum_{ij} J_{ij} S_i S_j$$

+ Ideally suited for QUBOs!

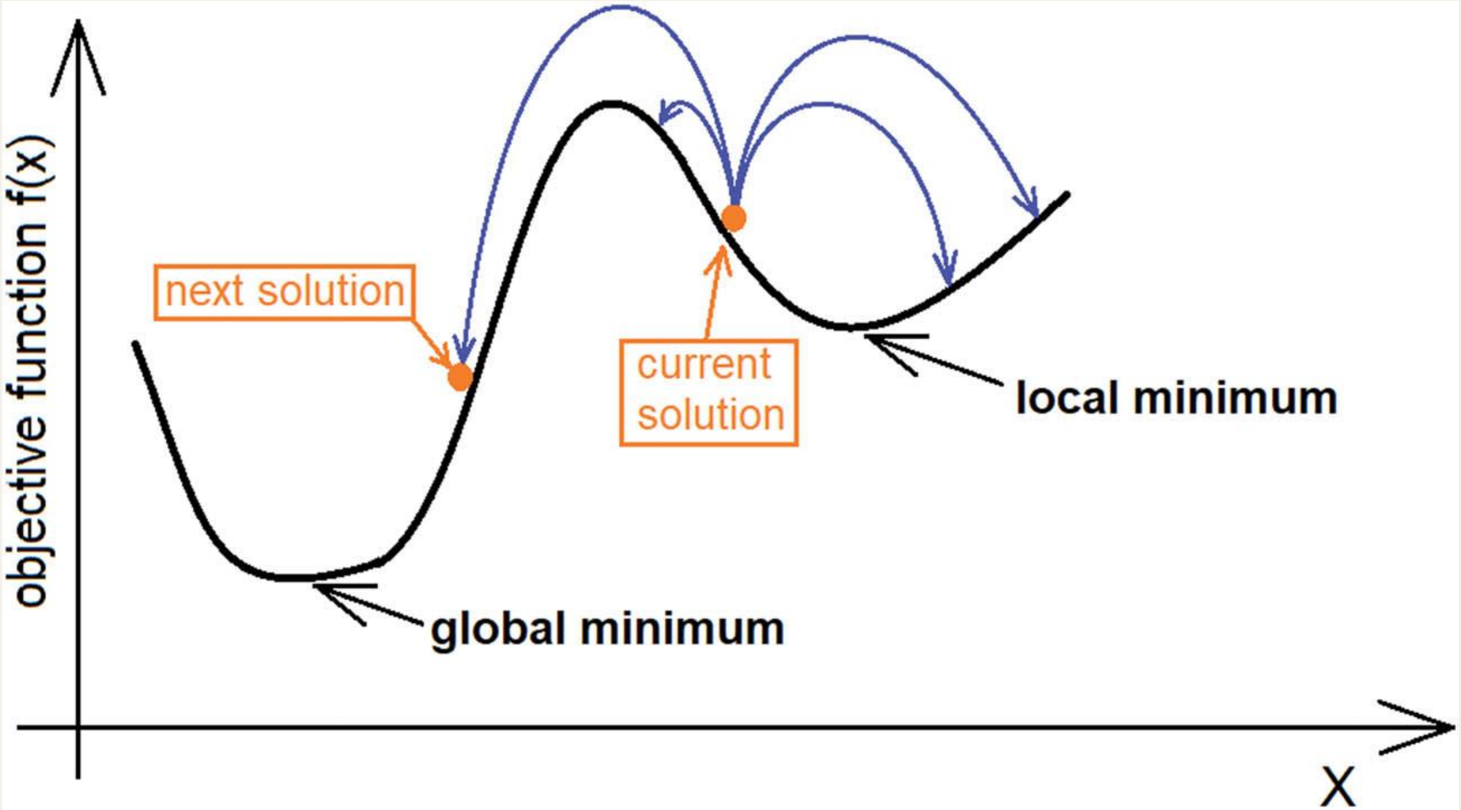
$$q_i = \frac{1 + S_i}{2}$$



Quantum annealing

+ Can be thought of as *analog quantum computation*
+ Named after simmulated annealing
+ Powered by the adiabatic theorem

Simulated annealing



Source: degruyter.com/document/doi/10.1515/geo-2020-0038/html

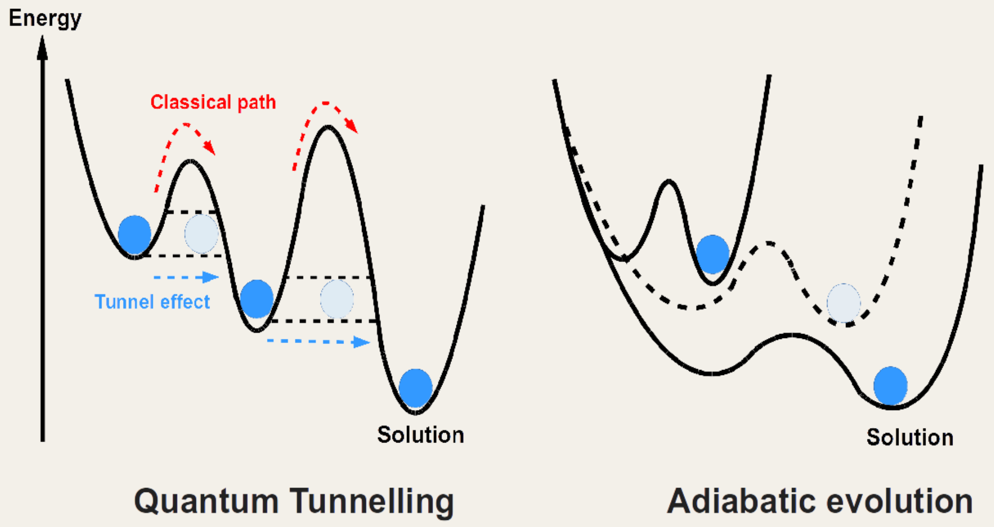
Quantum annealing – Adiabatic theorem

Start in ground state Ψ_0 and evolve under

$$H(0) \rightarrow H(T)$$

At time T you will be δ-close to ground state of $H(T)$ if

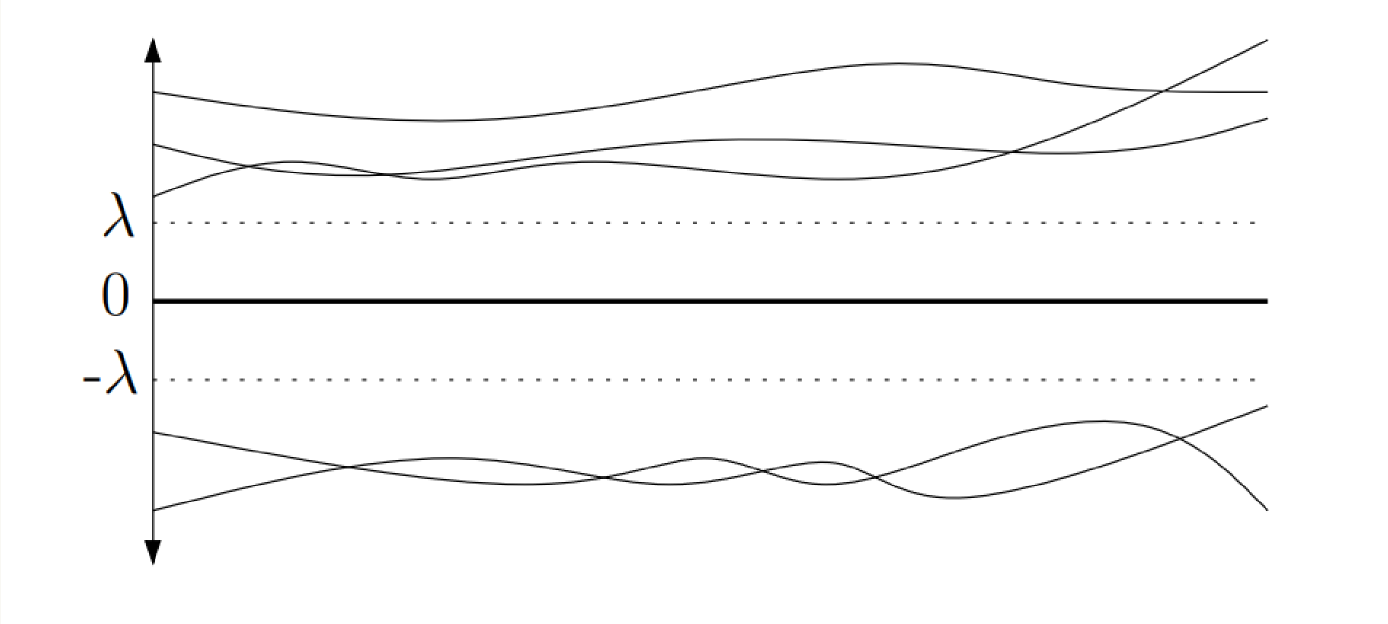
$$T \geq \frac{10^5}{\delta^2} \min\left\{\frac{\|\widetilde{H}'\|^4}{\lambda^4}, \frac{\|\widetilde{H}'\| \cdot \|\widetilde{H}''\|}{\lambda^3}\right\}$$



Source: medium.com/@quantum_wa

An Elementary Proof of the Quantum Adiabatic Theorem
A. Ambainis, O. Regev (2004)
arxiv.org/abs/quant-ph/0411152

Quantum annealing – Adiabatic theorem



An Elementary Proof of the Quantum Adiabatic Theorem
A. Ambainis, O. Regev (2004)
arxiv.org/abs/quant-ph/0411152

Some literature

PHYSICAL REVIEW E VOLUME 58, NUMBER 5 NOVEMBER 1998

Quantum annealing in the transverse Ising model

Tadashi Kadowaki and Hidetoshi Nishimori
Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152-8551, Japan
(Received 30 April 1998)

We introduce quantum fluctuations into the simulated annealing process of optimization problems, aiming at faster convergence to the optimal state. Quantum fluctuations cause transitions between states and thus play the same role as thermal fluctuations in the conventional approach. The idea is tested by the transverse Ising model, in which the transverse field is a function of time similar to the temperature in the conventional method. The goal is to find the ground state of the diagonal part of the Hamiltonian with high accuracy as quickly as possible. We have solved the time-dependent Schrödinger equation numerically for small size systems with various exchange interactions. Comparison with the results of the corresponding classical (thermal) method reveals that the quantum annealing leads to the ground state with much larger probability in almost all cases if we use the same annealing schedule. [S1063-651X(98)02910-9]

Quantum Annealing of a Disordered Spin System

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AT&T Bell Laboratories, 4 Independence Way, Princeton, New Jersey 07974

Abstract

Traditional simulated annealing utilizes thermal fluctuations for convergence in optimization problems. Quantum tunneling provides a different mechanism for moving between states, with the potential for reduced time scales. We compare thermal and quantum annealing in a model disordered Ising magnet, $LiHo_{0.44}Y_{0.56}F_4$, where the effects of quantum mechanics can be tuned in the laboratory by varying a magnetic field applied transverse to the Ising axis. The results indicate that quantum annealing indeed hastens convergence to the optimum state.

SCALABLE ARCHITECTURE FOR ADIABATIC QUANTUM COMPUTING OF NP-HARD PROBLEMS

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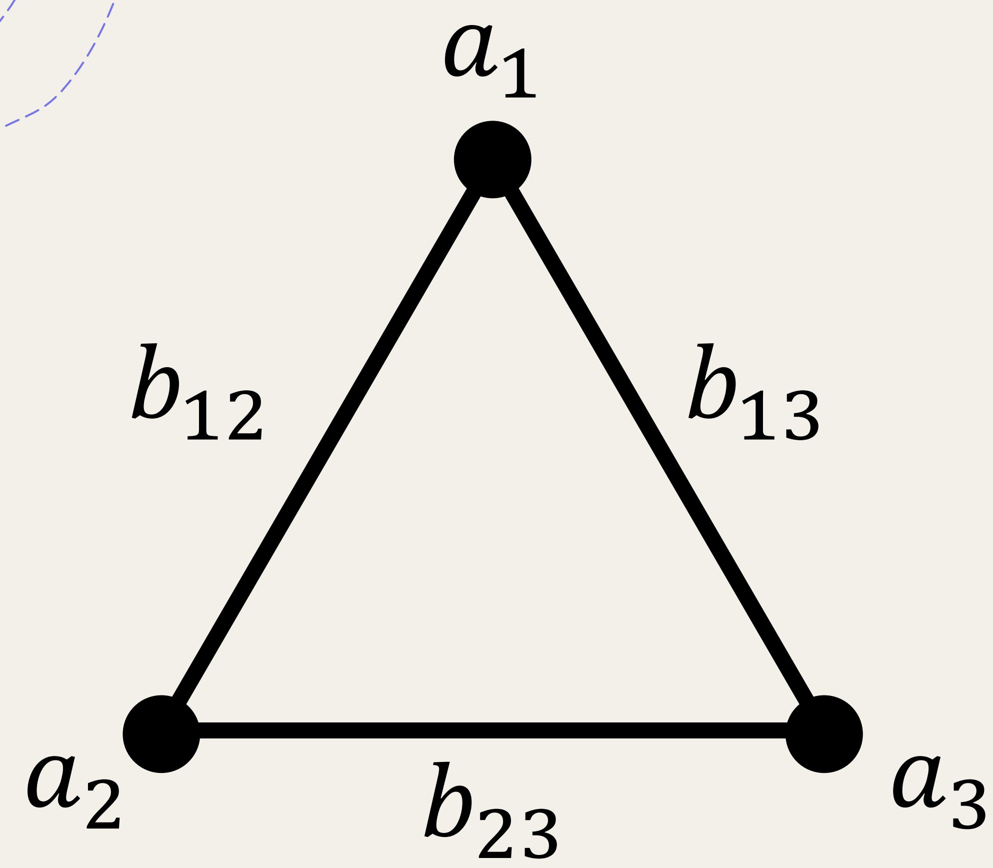
Abstract We present a comprehensive review of past research into adiabatic quantum computation and then propose a scalable architecture for an adiabatic quantum computer that can treat NP-hard problems without requiring local coherent operations. Instead, computation can be performed entirely by adiabatically varying a magnetic field applied to all the qubits simultaneously. Local (incoherent) operations are needed only for: (1) switching on or off certain pairwise, nearest-neighbor inductive couplings in order to set the problem to be solved and (2) measuring some subset of the qubits in order to obtain the answer to the problem.

QUBO Language



$$\begin{pmatrix} a_1 & b_{12} & b_{13} \\ 0 & a_2 & b_{23} \\ 0 & 0 & a_3 \end{pmatrix}$$

$$q^2 = \begin{cases} 1, & q = 1 \\ 0, & q = 0 \end{cases} \quad = q$$

QUBO Language

+ Impose equality

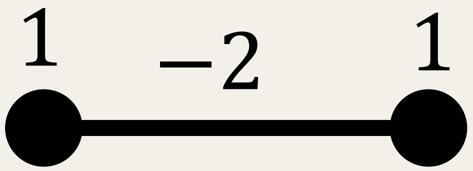
$$H = (q_i - c)^2 \quad \rightarrow \quad q_i = c$$

+ Only one value equal to 1

$$H = (\sum q_i - 1)^2 \quad \rightarrow \quad \exists!\, q_i = 1$$

+ Variable equality

$$H = \left(q_i - q_j\right)^2 \quad \rightarrow \quad q_i = q_j$$



QUBO problems

frontiers in PHYSICS

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Ising formulations of many NP problems

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We provide Ising formulations for many NP-complete and NP-hard problems, including all of Karp's 21 NP-complete problems. This collects and extends mappings to the Ising model from partitioning, covering, and satisfiability. In each case, the required number of spins is at most cubic in the size of the problem. This work may be useful in designing adiabatic quantum optimization algorithms.

Keywords: spin glasses, complexity theory, adiabatic quantum computation, NP, algorithms

1. INTRODUCTION

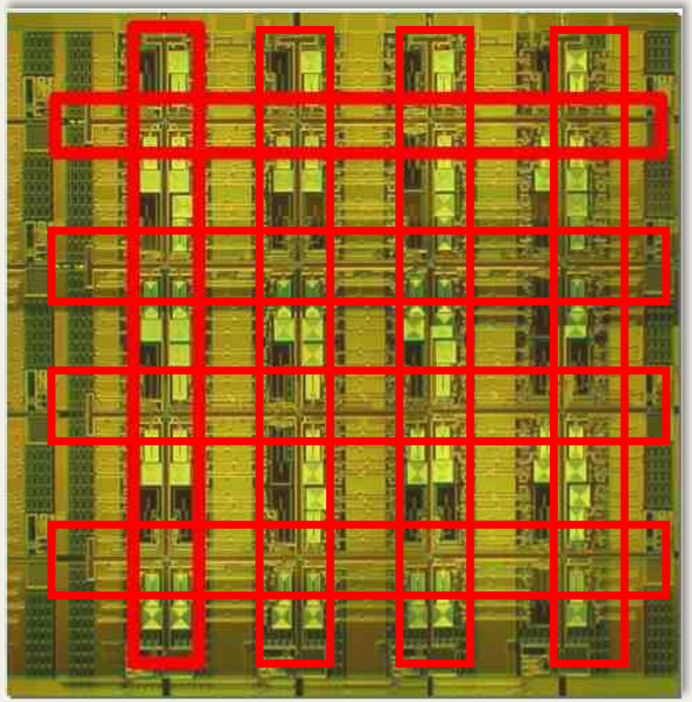
1.1. QUANTUM ADIABATIC OPTIMIZATION

Recently, there has been much interest in the possibility of using adiabatic quantum optimization (AQO) to solve NP-complete and NP-hard problems [1, 2] [1]. This is due to the following trick: suppose we have a quantum Hamiltonian H_P whose ground excited states [5] [2]. While it is unlikely that NP-complete problems can be solved in polynomial time by AQO, the coefficients α, β may be smaller than known classical algorithms, so there is still a possibility that an AQO algorithm may be more efficient than classical algorithms, on some classes of problems.

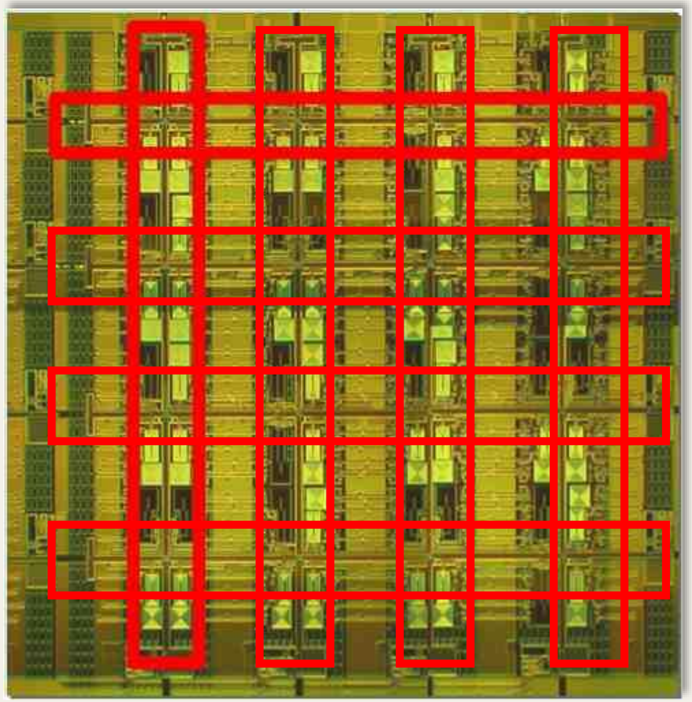
D-Wave Computer

D-Wave Computer

- Uses superconducting qubits, based on SQUIDS
- Each loop has quantized magnetic flux – analagous to spins
- Therefore, cryogenic
- We had cloud access to these computers



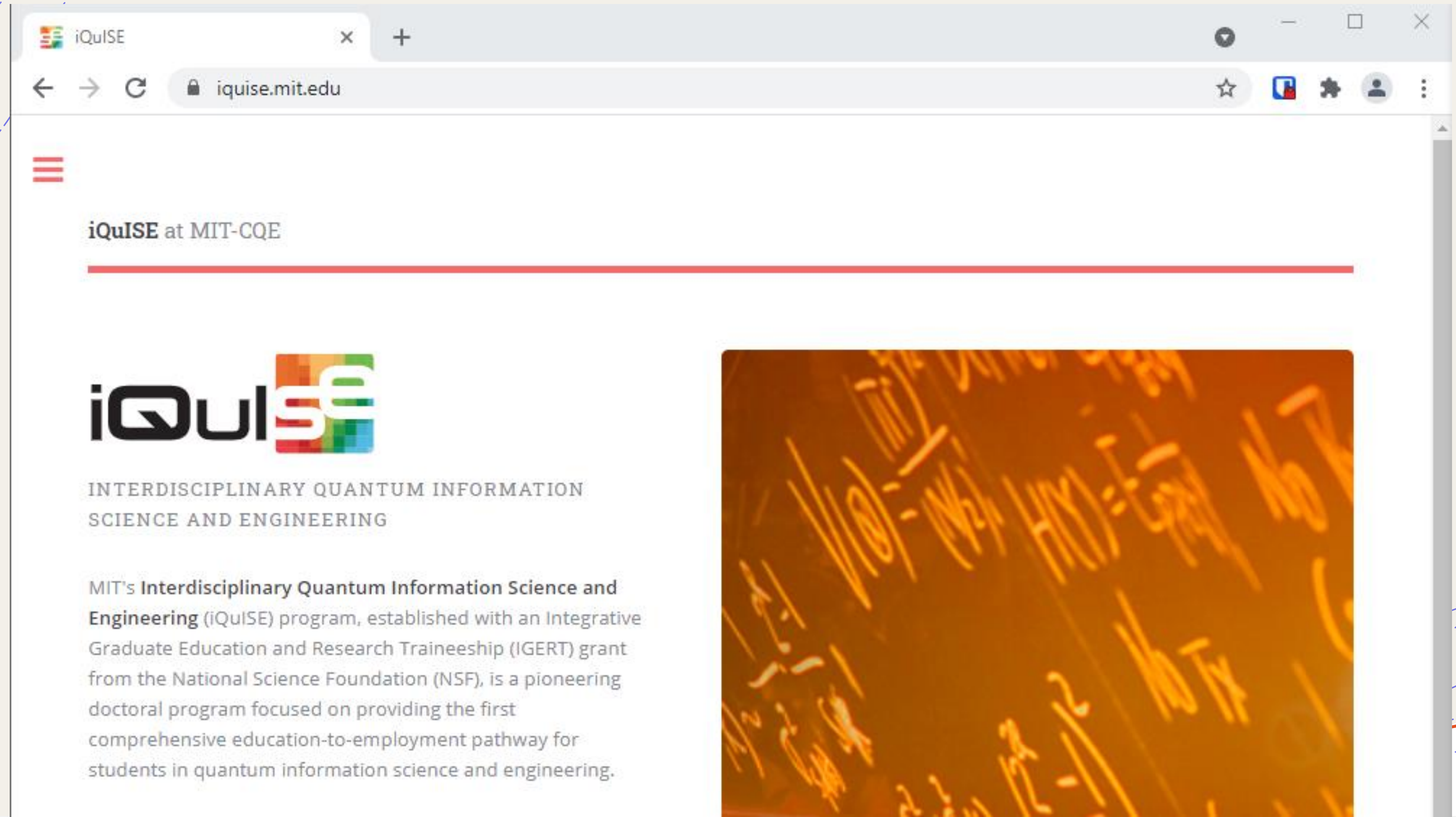
Does D-Wave have a QC?



iQuHACK Challenge

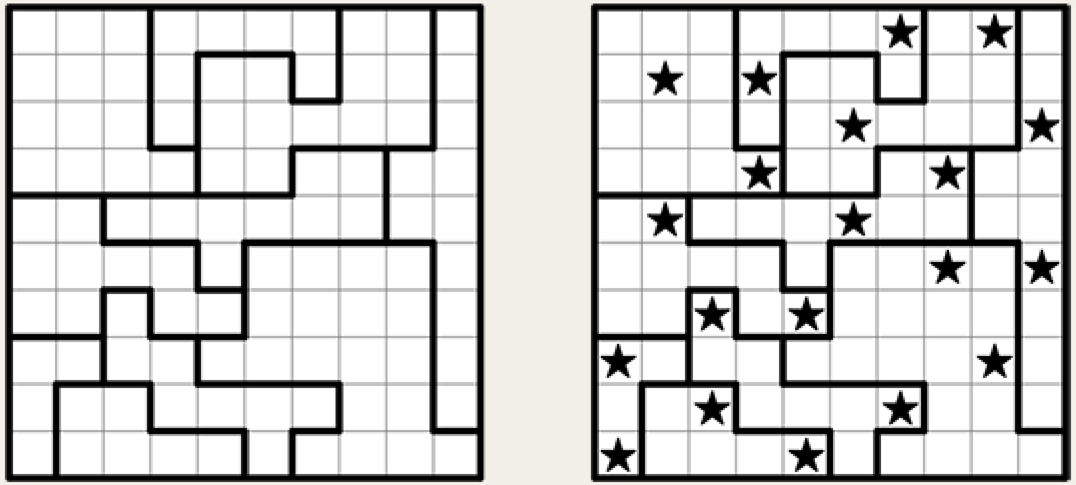
Battle star / Two-not-touch problem

About iQuHACK



Battle star / Two not touch

+ Regular puzzle at the *New York Times*
+ Rules:
  + Each row has 2 stars
  + Each column has 2 stars
  + Each region has 2 stars
  + All stars are neighbour-free



The challenge

Given a puzzle…

```
[ ]  grid = np.array([
         [0, 1, 1, 1, 1],
         [0, 1, 1, 1, 2],
         [0, 0, 0, 2, 2],
         [3, 0, 3, 2, 2],
         [3, 3, 3, 4, 4],
     ])
```

…find a solution

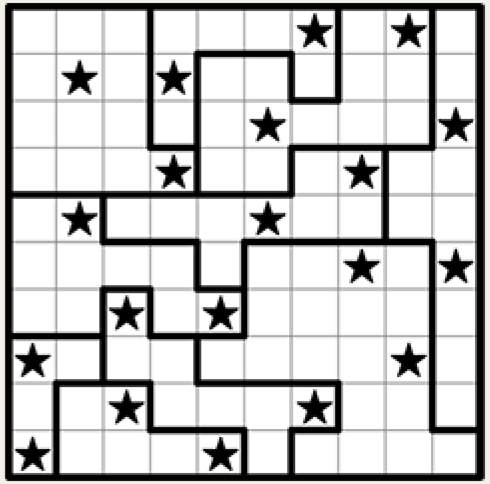
```
[ ]  solution = np.array([
         [0, 1, 0, 0, 0],
         [0, 0, 0, 0, 1],
         [0, 0, 1, 0, 0],
         [1, 0, 0, 0, 0],
         [0, 0, 0, 1, 0],
     ])
```

Modeling battlestar

Problem variables

+ $N \times N$ grid
+ Each cell is a variable x_{ij}:

$$x_{ij} = \begin{cases} 1, & \text{if } (i,j) \text{ has a star} \\ 0, & \text{otherwise} \end{cases}$$

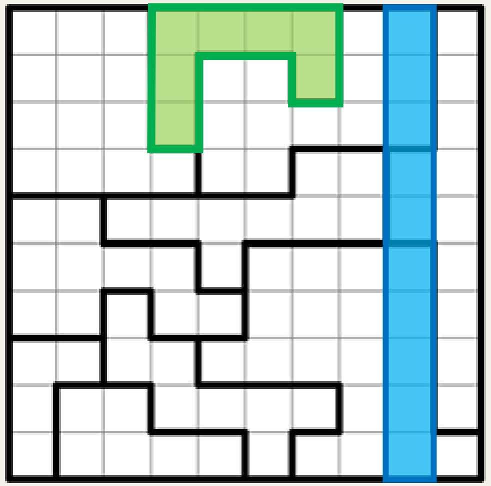


Modeling battlestar

Region contraints

- Lines, columns and regions are similar
- Let a region be $R = \{P_1, \ldots, P_r\}$
- Suppose we want S stars in each region
- The constraint hamiltonian is

$$H = \left(S - \sum_{P \in R} x_P\right)^2$$



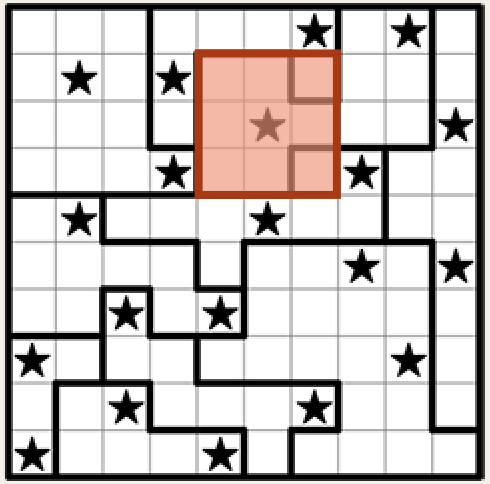
Modeling battlestar

Nearest neighbours

+ We want to penalize close neighbours
+ Note that

x_P	x_Q	$x_P x_Q$
0	0	0
0	1	0
1	0	0
1	1	1

$$\Longrightarrow H = \sum_{\langle P, Q \rangle} x_P x_Q$$



Modeling battlestar

The full Hamiltonian

- Our QUBO Hamiltonian:

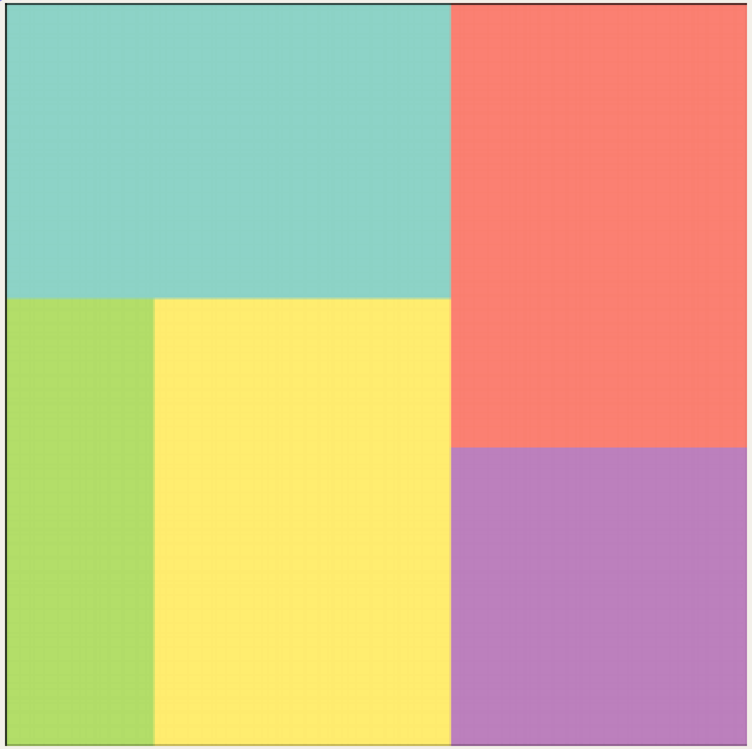
$$H = \gamma_r \underbrace{\sum_R \left(S - \sum_{P \in R} x_P \right)^2}_{\text{Regions}} + \gamma_n \underbrace{\sum_{\langle P,Q \rangle} x_P x_Q}_{\text{Neighbours}}$$

Coding the solution

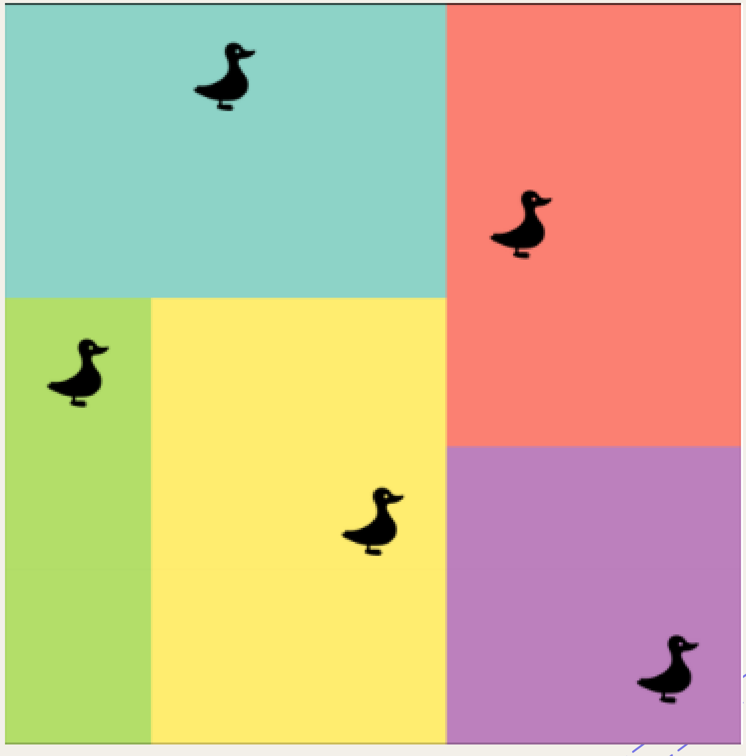
+ github.com/spratapsi/battle_star

Results for $N = 5$

Input region

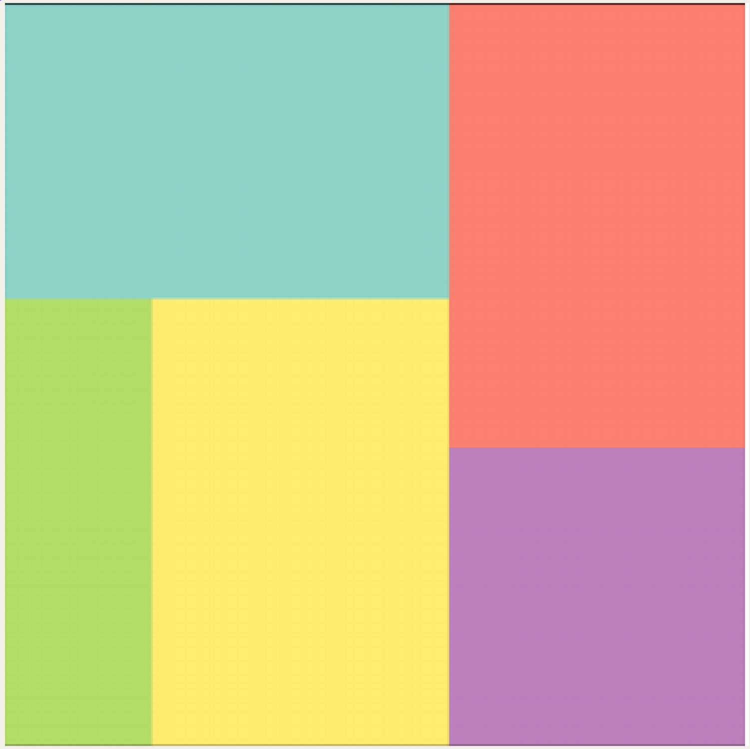


Output

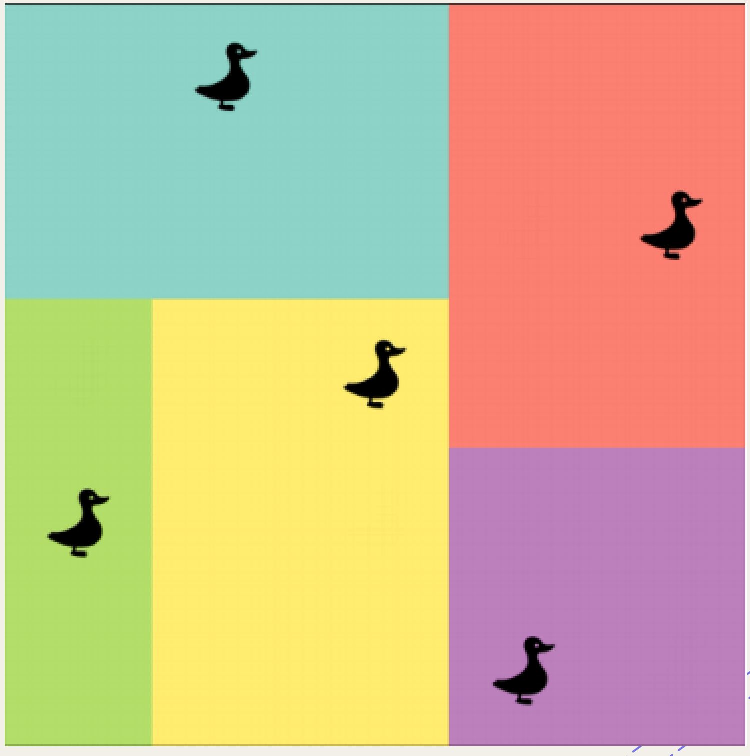


Results for $N = 5$

Input region

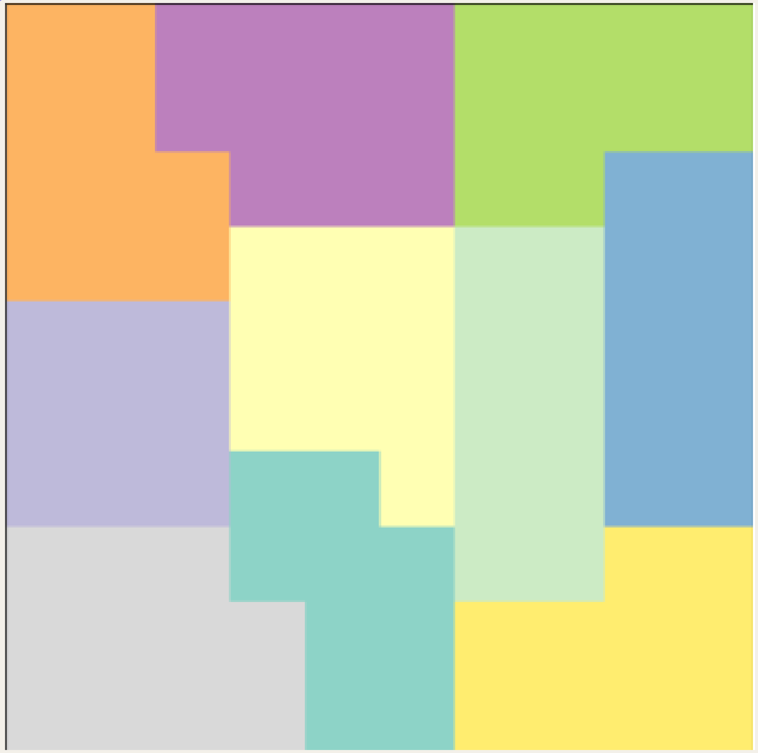


Output

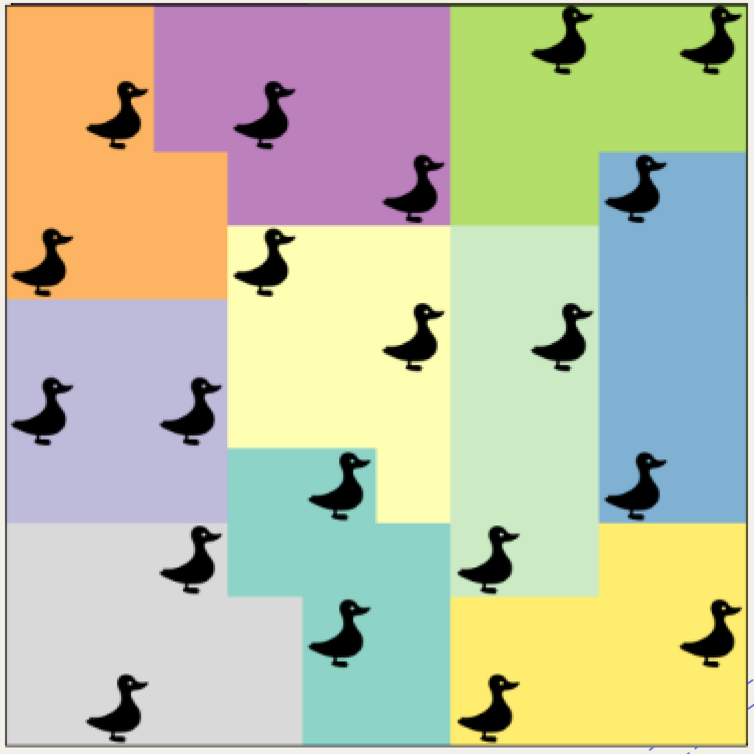


Results for $N = 10$

Input region

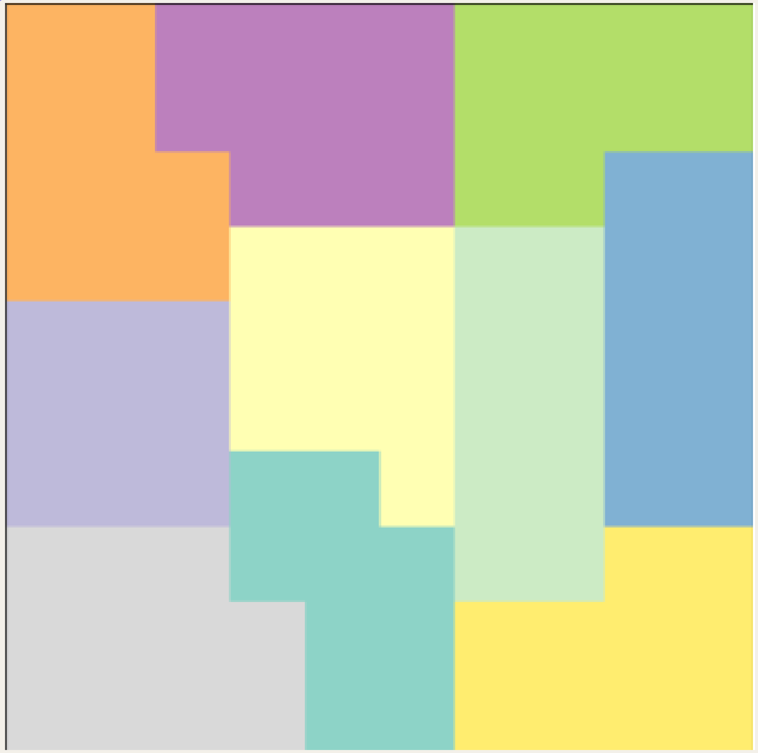


Output

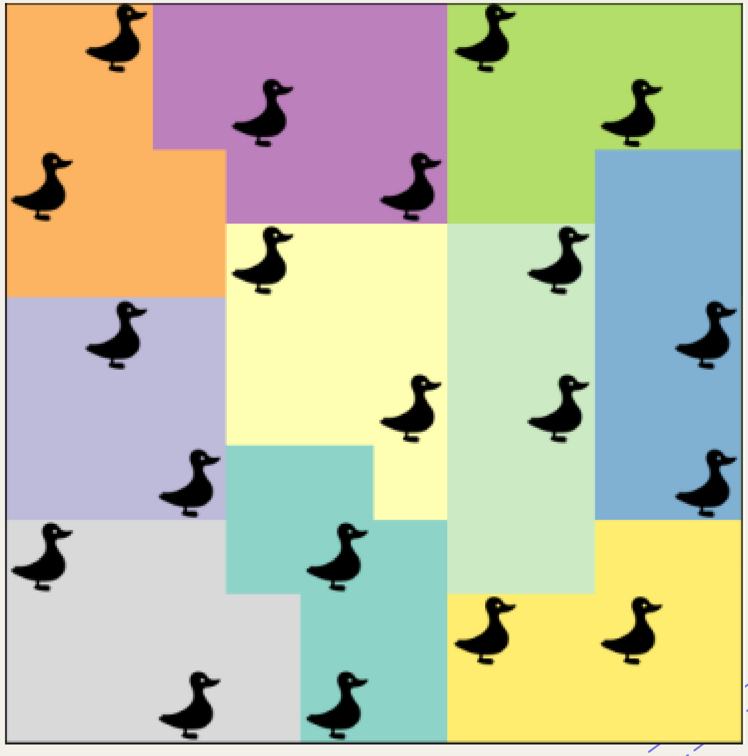


Results for $N = 10$

Input region



Output



Generating puzzles

+ What if we could *generate* puzzles as well?
+ Given: grid size (N) and # of stars (S)
+ Strategy:
    1. Generate stars in empty grid
    2. Create *compatible* regions
+ New variables x_P^R for region generation

Generating puzzles

Create star distribution

- This part is done!
- Use same Hamiltonian as before, without regions

$$H = \gamma_r \underbrace{\sum_{R} \left(S - \sum_{P \in R} x_P \right)^2}_{\text{Rows and columns}} + \gamma_n \underbrace{\sum_{\langle P, Q \rangle} x_P x_Q}_{\text{Neighbours}}$$

Generating puzzles

Create regions

+ Constraint 1: Each cell must have a unique region

$$H = \left(1 - \sum_{R} x_P^R\right)^2, \qquad \text{for each } P$$

Generating puzzles

Create regions

+ Constraint 2: Each region has S stars

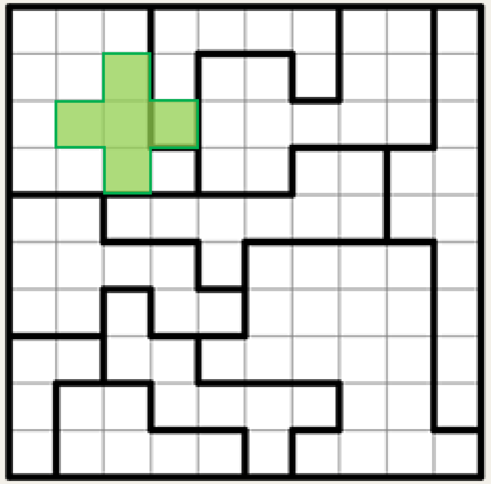
$$H = \left(1 - \sum_{P \in stars} x_P^R\right)^2, \qquad \text{for each } R$$

Generating puzzles

Create regions

- Constraint 3: connectedness

$$H = -\sum_{(P,Q)} x_P^R x_Q^R, \qquad \text{for each } R$$



- However, this favors giant regions!

Generating puzzles

Create regions

- Constraint 4: connectedness

$$H = \left(N - \sum_{P} x_P^R\right)^2, \qquad \text{for each } R$$

- Every region has the same size. Notice we are minimizing:

$$H = \frac{1}{N}\sum_{R}\left(\sum_{P} x_P^R - N\right)^2, \qquad \text{for each } R$$

Generating puzzles

Create regions

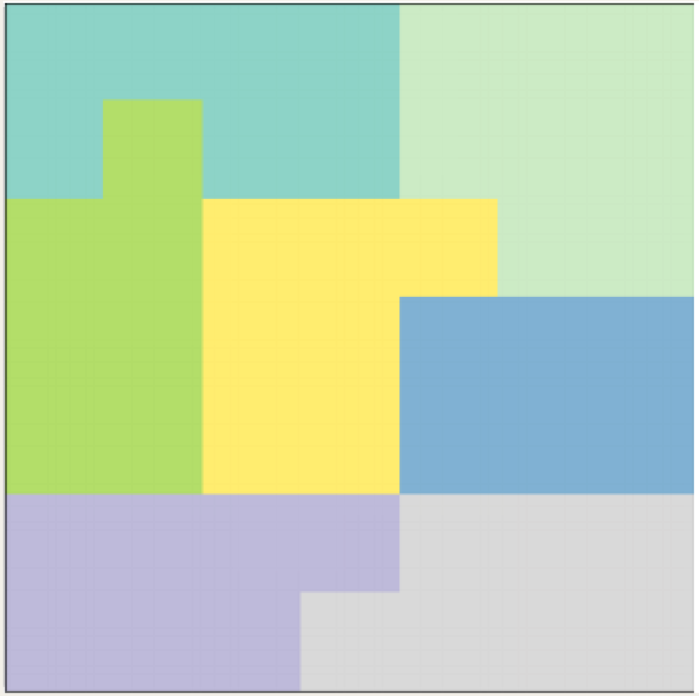
+ Full Hamiltonian

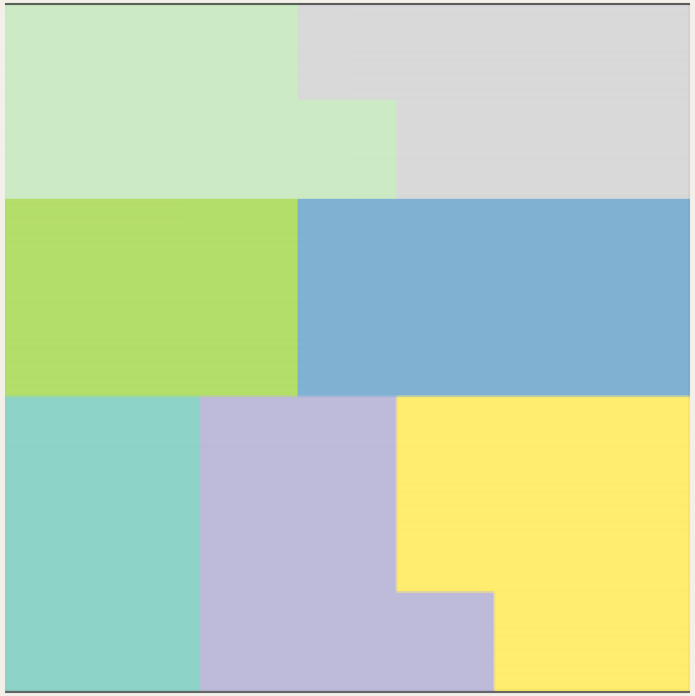
$$H = \gamma_1 \sum_P \left(1 - \sum_R x_P^R\right)^2 + \gamma_2 \sum_R \left(1 - \sum_{P \in stars} x_P^R\right)^2$$

$$-\gamma_3 \sum_R \sum_{(P,Q)} x_P^R x_Q^R + \gamma_4 \sum_R \left(\sum_P x_P^R - N\right)^2$$

Generating puzzles

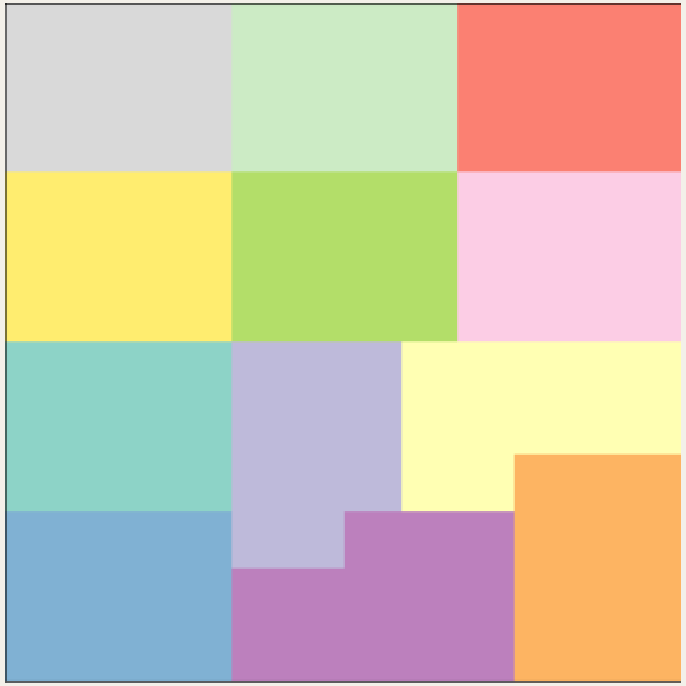
Results for $N = 7$

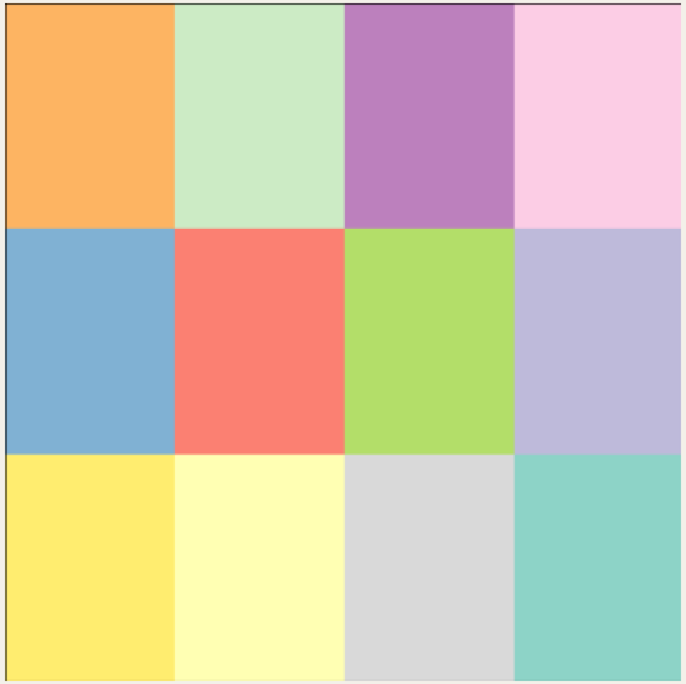




Generating puzzles

Results for $N = 12$





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