Hadron Structure from Nonperturbative QCD

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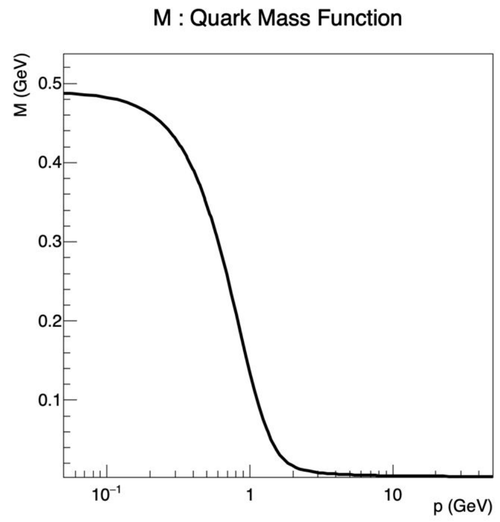
Quark Propagator - A Fundamental Quantity

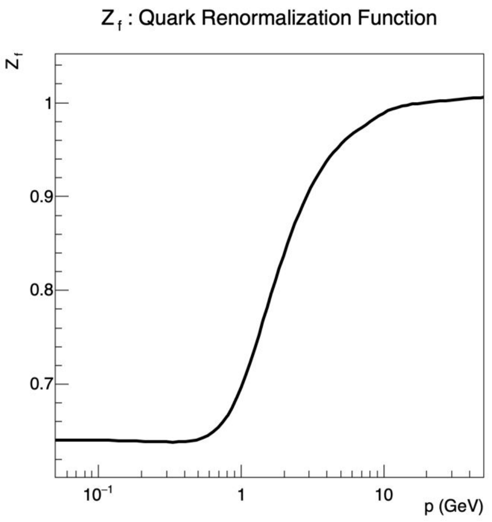
Define S dressing functions ⇔ Define its components

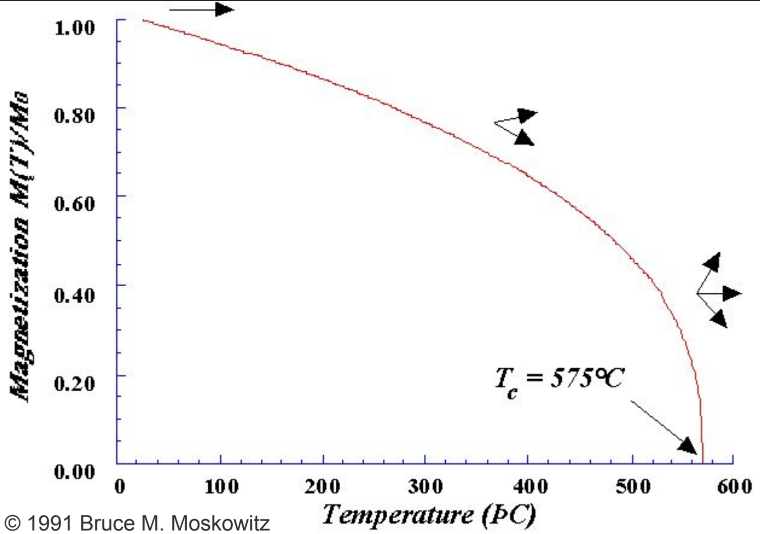
Functions of p^2

$$S = -i\not{p}\,\sigma_v + \sigma_s = \frac{Z_f}{p^2+M^2}\left(-i\not{p} + M\right)$$

One can define different sets of 2 dressing functions





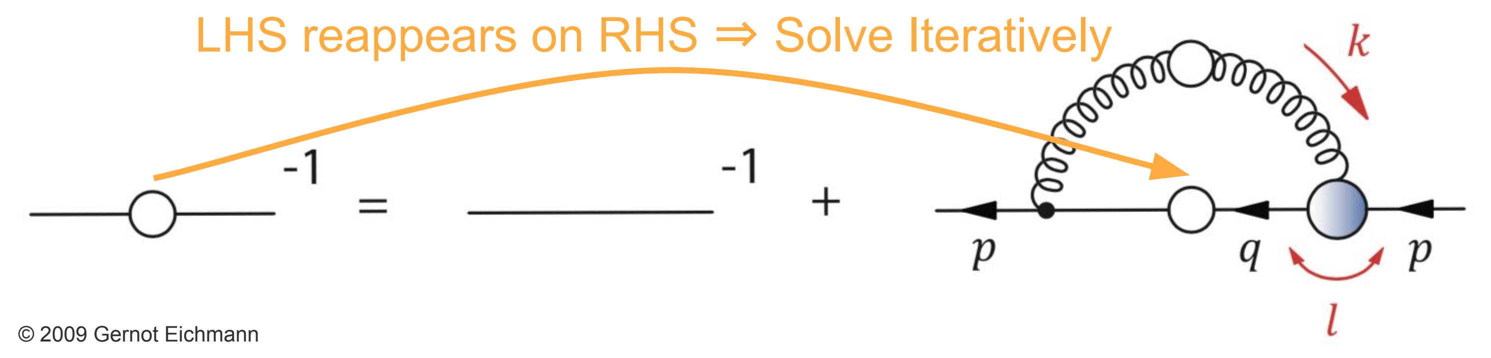


Dyson-Schwinger Equation

Components: What to extract from S

DSE: How to get S

$$S^{-1} = Z_2(i\not{p} + M_\Lambda) + \Sigma$$



DSE in a nutshell: All the ways a quark can emit and absorb gluons.

$Z_2 = Z_f^{-1}|_{p2=\Lambda 2}$ $M_\Lambda = M|_{p2=\Lambda 2}$ Λ : Just a Scale Factor

Σ : Quark Self-Energy → We used a model of the gluons, instead of a gluon DSE.

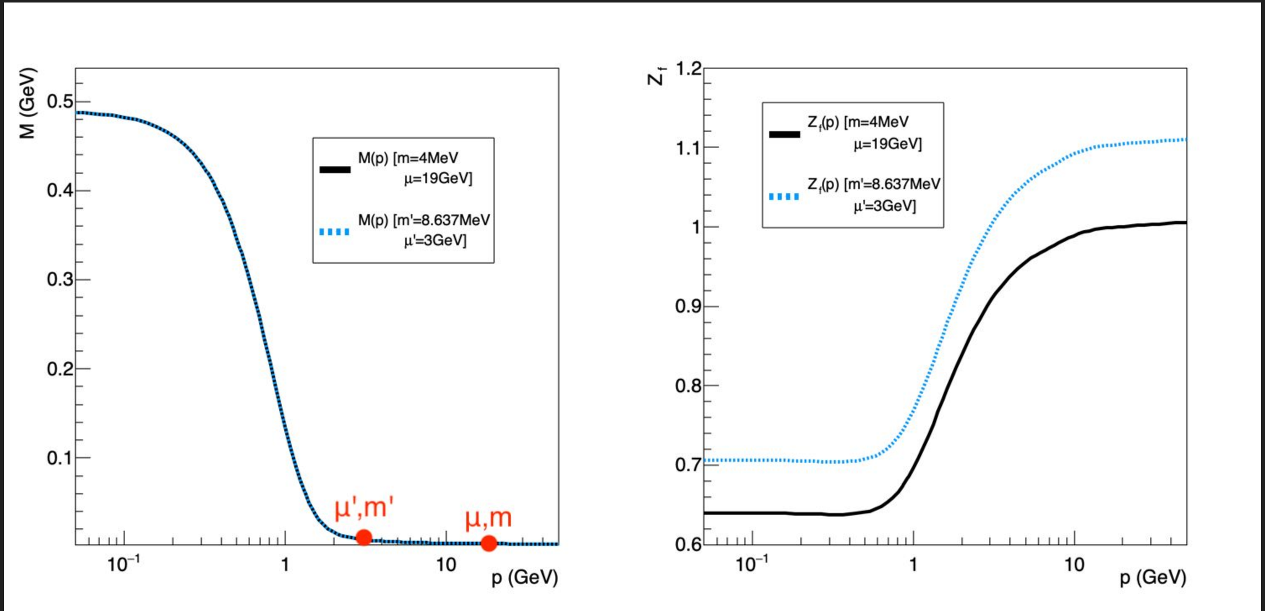
$p^2 \in \mathbb{R}$

Fortran90 (for Computation) + ROOT (for Graphics)

Iterate

① Calculate the Σ integrals → Update the M and Z_f functions

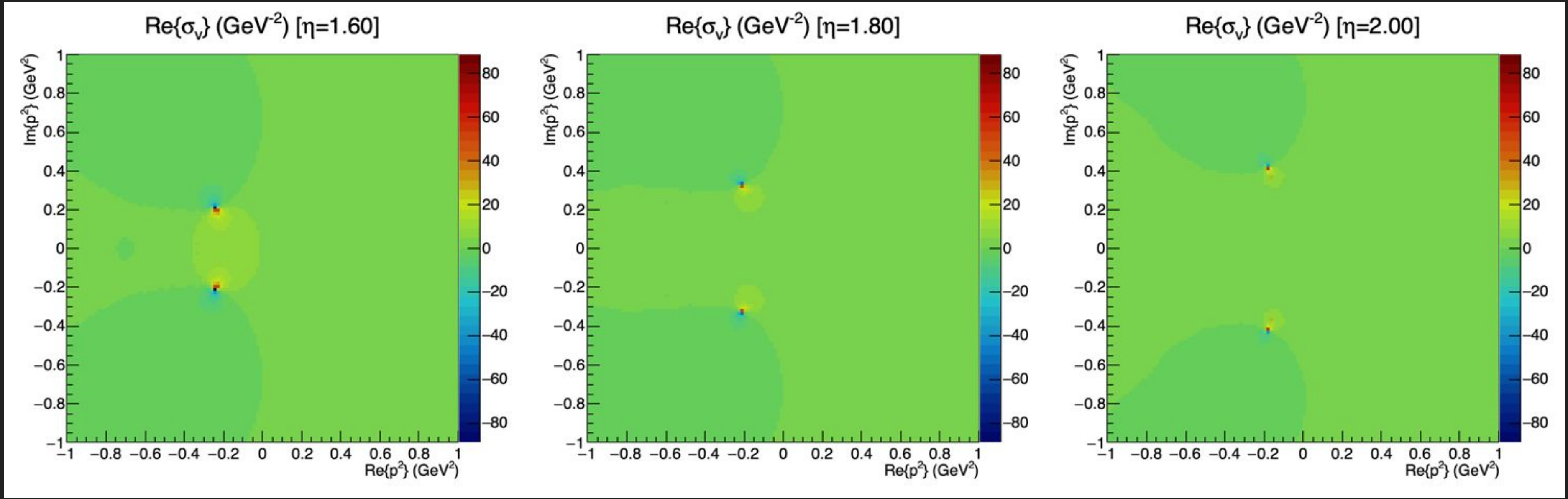
② Affirm Input Point ($p=\mu$, $M|_\mu=m$, $Z_f|_\mu=1$) → Update the Z_2 and M_Λ numbers



Renormalization-Group Invariance for $M(p^2)$, but not for $Z_f(p^2,\mu^2)$

$p^2 \in \mathbb{C}$

❶ Converge for $p^2 \in \mathbb{R}$ ❷ Calculate M and Z_f through the DSE, now with $p^2 \in \mathbb{C}$



The poles move around with η (a parameter used in Σ)

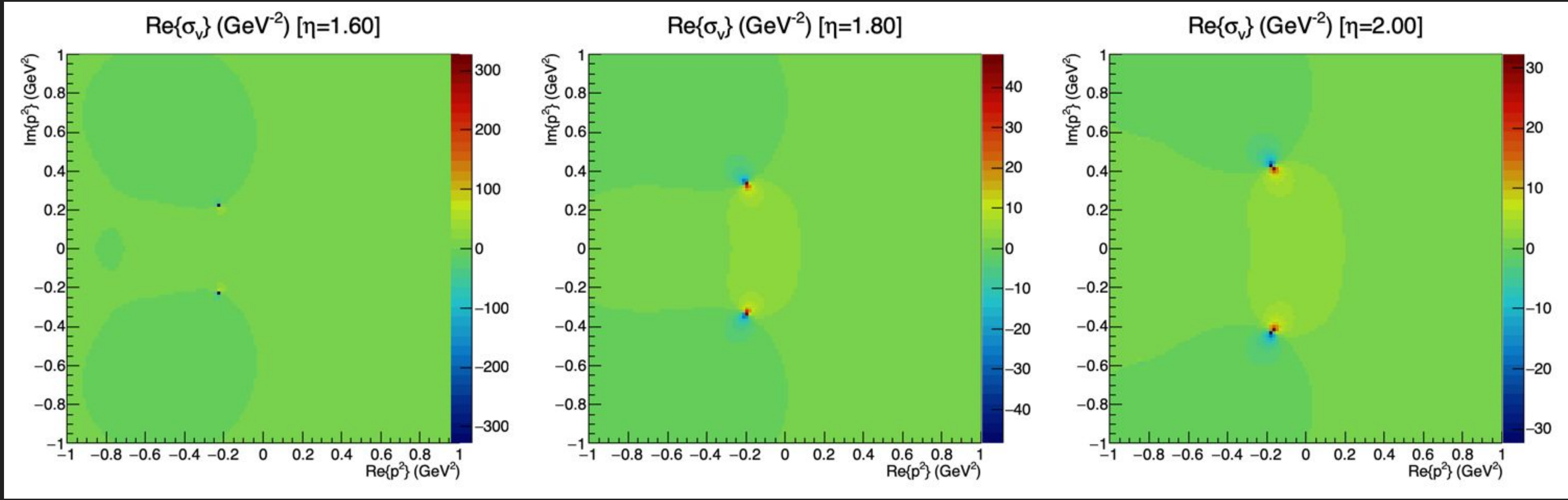
Free Particle ⇒ Pole $\in \mathbb{R}^-$

Get $f(p^2 \in \mathbb{C})$ from $f(p^2 \in \mathbb{R})$

Other models of Σ where a direct substitution $p^2 \in \mathbb{C}$ is impossible

Schlessinger Point Method (with n input points):

$$f(z) = \cfrac{c_1}{1 + \cfrac{c_2(z - z_1)}{1 + \cfrac{c_3(z - z_2)}{\cdots}}}$$



One can get the position and residue of each pole

Thank you

What’s next?

- Schlessinger intensive test
- Other models for Σ
- Machine Learning as alternative to Schlessinger

From Light-Front Wave Functions to PDFs

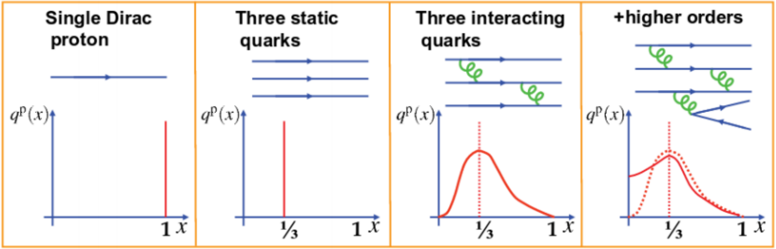
Eduardo Ferreira - Gernot Eichmann

LIP Internship Programme 2020

September 11, 2020

On the subject of PDFs

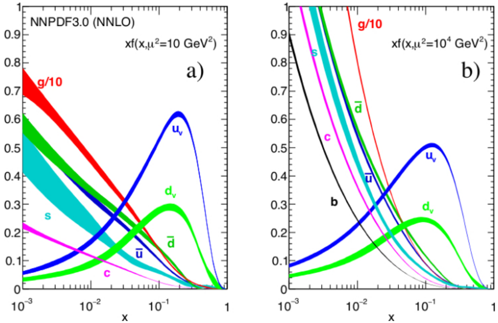
- Parton Distribution Functions (PDFs) give us the fraction of momentum carried by each constituent in a hadron
- QCD and its interactions give these functions a interesting albeit mathematically difficult nature



Taken from Márcia Quaresma's talk on the *Initial Lectures and Tutorials Week*

Why PDFs?

- They are universal
- PDFs are extremely useful in the calculation of other quantities:
  - Determination of the background for experiments
  - Search for new physics at LHC
  - Predictions of the SM
- Current understanding comes from fits from experimental results



Calculate PDFs theoretically

- The goal of our work is to calculate PDFs directly from QCD
  1. Calculate the solutions of the Bethe-Salpeter equation via the Nakanishi Integral Representation
  2. Calculate the Light Front Wave Function of the system (where distances are light-like $x^2 = 0$)
  3. Calculate the PDFs via the calculated Light Front Wave Functions [1]
- We need to implement numerical methods to solve these equations
- Singularity structure of the equations may require the use of clever integration paths - **contour deformations**

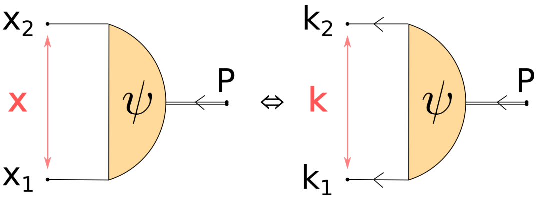
[1] See Leitao et al., Eur. Phys. J C 77 for a comparison of other methods

Bethe-Salpeter Wave Function

- Let $|0\rangle$ be the vaccum and $|P\rangle$ a 2 body bound state of total momentum P
- $\psi\left(x,P\right)$ is the Bethe-Salpeter Wave Function, defined via:

$$\psi(x,P) = \langle 0|\, \mathrm{T}\phi(0)\phi(x)\, |P\rangle \tag{1}$$

$$\psi(k,P) = \int d^4x e^{-ik\cdot x}\psi(x,P) \tag{2}$$

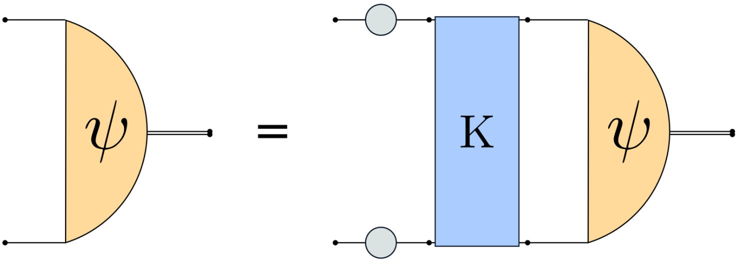


Bethe-Salpeter Equation

- To calculate $\psi(k,P)$ we use the Bethe-Salpeter Equation (for the amplitude) [2]

$$\psi(k,P) = \int \frac{d^4q}{(2\pi)^4} K(k,q) G_0(q,P) \psi(q,P) \tag{3}$$

$K(k,q)$ contains all irreducible diagrams; $G_0(q,P)$ are the particles' propagators



[2] See Eichmann, G. and Duarte, P. and Peña, M. T. and Stadler, A., Phys. Rev. D 100 (2019) 9, 094001

Bethe-Salpeter Equation

Nakanishi Representation

- We are using a simple model - scalar particles with scalar interaction: $\mathcal{L} = g\phi\phi\chi$
- T. Frederico; G. Salmè and M. Viviani [3] propose a different way of solving the equation:

$$\psi(x,z) = \frac{1}{m^4}\int_0^\infty dx' \int_{-1}^{1} dz' \frac{(1-z'^2)h(x',z')}{\left[x+x'+1+t+\sqrt{xt}zz'\right]^3}$$

$h(x,z)$ is the Nakanishi Weight Function, $t = \frac{P^2}{4m^2}$
The equation for determining $h(x,z)$ (after discretizing):

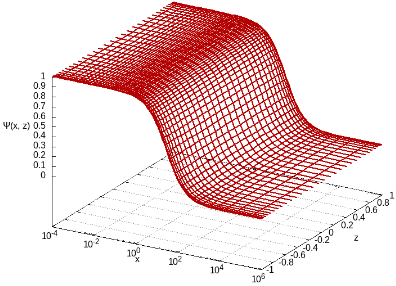
$$\lambda \mathbf{B}\mathbf{h} = \mathbf{K}\mathbf{h} \leftrightarrow \lambda \mathbf{h} = \mathbf{B}^{-1}\mathbf{K}\mathbf{h}$$

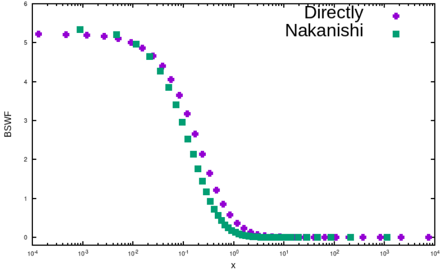
[3] Phys. Rev. D 89 (2014) 1, 016010

BS Wave Function

Results

- Results obtained using our own implementation of the equations
- Calculated with $\sqrt{t} = 0.75i$, $\beta = \frac{\mu}{m} = 4$





- Note: Small angular (z) dependency. Only radial dependency (on x)

Light-Front Wave Function

- This is just the BSWF at light-like distances $x^2 = 0$. For example: $x = \lambda n$

$$n = (0, 0, 1, -i)^T \implies n^2 = 0 \implies x^2 = 0 \qquad (4)$$

  $k^+ = k.n$ is the momentum along the x direction
- We can redefine the momentum variables for both constituents:
  - $k_1^+ = \xi P^+$, where $0 < \xi < 1$
  - $k_2^+ = (1 - \xi) P^+$ since $k_2 = P - k_1$
  - $k = \frac{k_1 - k_2}{2} \implies k^+ = (\xi - \frac{1}{2}) P^+ \equiv \frac{-\alpha}{2} P^+$

    Work with Light-Front WF using the variable $\alpha \implies \psi(\alpha)$.

Light-Front Wave Function

- Starting with the Fourier Transform of the Bethe-Salpeter WF:

$$\psi(\lambda n, P) = \int \frac{d^4k}{(2\pi)^4} e^{i\lambda k\cdot n} \psi(k, P) \tag{5}$$

Define the LFWF $\psi(\alpha)$:

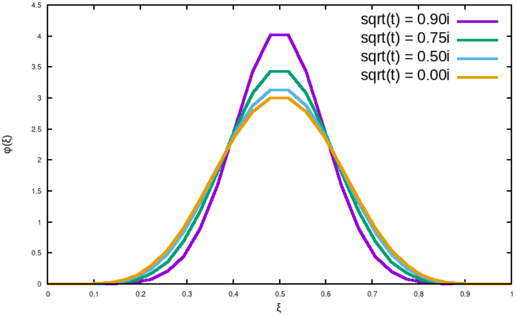
$$\psi(\alpha) \equiv \int \frac{d\lambda}{4\pi} e^{i\frac{\alpha}{2}P\cdot n\lambda} \psi(\lambda n, P) \tag{6}$$

$$= \frac{1}{P\cdot n} \int \frac{d^4k}{(2\pi)^4} \psi(k, P) \delta\left(\alpha + \frac{2k\cdot n}{P\cdot n}\right) \tag{7}$$

As $d^4k = \frac{1}{2} d^2k_\perp dk^+ dk^-$, this is:

$$\int d^2k_\perp \int dk^- \psi(k, P)\Big|_{k^+ = \frac{-\alpha}{2}P^+} \tag{8}$$

- Via the Nakanishi Representation using $\beta = 0.50$



- Both constituents sharing half the total momentum is the most common form.

What's next?

- Get the LFWF without the Nakanishi Representation. This requires **contour deformations** because of the singularity structure.
  Example: Ansatz for the BSWF:

$$\frac{1}{(x+t+1)^2 - 4xtz^2} \frac{1}{x+\gamma} \tag{9}$$

- Get the desired PDF's from the calculated LFWF

BACKUP

Bethe-Salpeter Equation

For our model, one can write the BS Equation for the Nakanishi Representation as:

$$\int_0^\infty dx' \frac{h(x',z)}{[x' + \mathcal{N}(x,z)]^2} = \int_0^\infty dx' \int_{-1}^{1} dz' V(x,z,x',z') \frac{1-z'^2}{1-z^2} h(x',z')$$

$$V(x,x',z,z') = \frac{c}{2} \frac{1}{\mathcal{N}(x,z)} \int_0^1 dv \left[K(v,x,x',z,z') + K(v,x,x',-z,-z') \right.$$

$K(v,x,x',z,z')$ is defined as:

$$\frac{\theta(z'-z)(1+z)^2 v^2}{[v(1-v)(1+z')\mathcal{N}(x,z) + v^2(1+z)\mathcal{N}(x',z') + (1-v)(1+z)(\beta + vx')}$$

$$\beta = \frac{\mu}{m};\; c = \frac{g^2}{(4\pi m)^2};\; \mathcal{N}(x,z) = x + 1 + t(1-z^2)$$

- To solve the BS Equation, we need to do the integrals numerically:
  - Gaussian Quadrature

$$\int_a^b dx\, w\left(x\right) f\left(x\right) \approx \sum_{i=1}^{i=N} w_i\, f\left(x_i\right) \tag{10}$$

  - x_i and w_i are determined by the set of orthogonal polynomials w.r.t the inner product $\langle f, g\rangle = \int_a^b dx\, w(x) f(x) g(x)$
  - x_i are the roots of the polynomial of degree n
  - $w_i = \frac{\langle P_{N-1}, P_{N-1}\rangle}{P_{N-1}(x_i) P_N'(x_i)}$
- Different $w(x)$ will select the orthogonal polynomials:
  - $w(x) = 1$ - Legendre Polynomials ($x \in [-1, 1]$)
  - $w(x) = \sqrt{1 - x^2}$ - Chebyshev-U (2nd Kind) Polynomials ($x \in [-1, 1]$)

- We discretize the problem with N_x points for x and N_z points for z
- Expand the z dependence in a orthogonal basis:
  - Chebyshev U polynomials $Y_m(z)$ such that $\frac{2}{\pi}\int_{-1}^{1} dz\sqrt{1-z^2}Y_m(z)Y_n(z) = \delta_{nm}$
  - Using gegenbauer Polynomials $G_l(z) = 4(1-z^2)\Gamma(\frac{5}{2})\sqrt{\frac{(2l+5/2)(2l)!}{\pi\Gamma(2l+5)}}C_{2l}^{5/2}(z)$, with $\int_{-1}^{1} dz G_j(z)G_i(z) = \delta_{ij}$
- Write the equation in matrix form:

$$\lambda\mathbf{Bh} = \mathbf{Kh} \implies \mathbf{B}^{-1}\mathbf{Kh} = \lambda\mathbf{h} \tag{11}$$

This is now a standard eigenvalue equation

Light-Front Wave Function

- Starting with the Fourier Transform of the Bethe-Salpeter WF:

$$\psi(\lambda n, P) = \int \frac{d^4k}{(2\pi)^4} e^{i\lambda k\cdot n}\psi(k, P) \tag{12}$$

Define the LFWF $\psi(\alpha)$:

$$\psi(\alpha) \equiv \int \frac{d\lambda}{4\pi} e^{i\frac{\alpha}{2}P\cdot n\lambda}\psi(\lambda n, P) \tag{13}$$

$$= \int \frac{d^4k}{(2\pi)^4} \int \frac{d\lambda}{4\pi} e^{i\lambda(k\cdot n + \frac{\alpha}{2}P\cdot n)}\psi(k, P) = \frac{1}{P\cdot n} \tag{14}$$

As $d^4k = \frac{1}{2}d^2k_\perp dk^+ dk^-$, this is:

$$\int d^2k_\perp \int dk^- \psi(k, P)\Big|_{k^+ = \frac{-\alpha}{2}P^+} \tag{15}$$