Hyperbolicity of General Relativity in Bondi-like gauges

Thanasis Giannakopoulos

Instituto Superior Técnico

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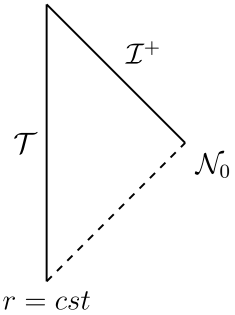
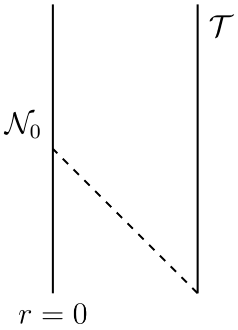
Spacetime foliated with null hypersurfaces.

Asymptotically flat
(Bondi-Sachs)

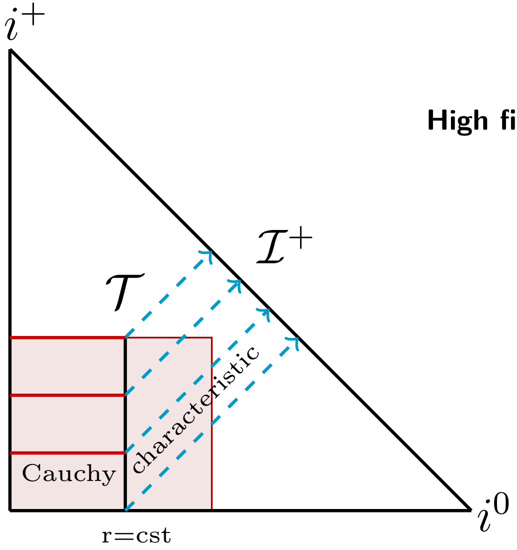


Gravitational waves

Asymptotically AdS
(Affine-null)



Numerical holography



High fidelity waveforms

Cauchy-Characteristic extraction

A numerical solution can converge to the continuum one only for well-posed PDE problems.

Well-posedness: property of a Partial Differential Equation (PDE) problem

- There exists a unique solution
- It depends continuously on the given data, in some norm

In preparation with David Hilditch & Miguel Zilhão

Well-posedness & hyperbolic PDEs

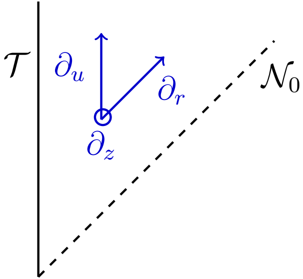
Hyperbolic PDE system: $\mathbf{A}^t \, \partial_t \mathbf{u} + \mathbf{A}^p \, \partial_p \mathbf{u} = \mathbf{S}\,,$

$$\mathbf{u} = (u_1, u_2, \ldots, u_q)^T\,, \quad \mathbf{A}^\mu = \begin{pmatrix} a^\mu_{11} & \ldots & a^\mu_{1q} \\ \vdots & \ddots & \vdots \\ a^\mu_{q1} & \ldots & a^\mu_{qq} \end{pmatrix}\,, \quad det(\mathbf{A}^t) \neq 0\,.$$

Principal symbol: $\mathbf{P}^s = (\mathbf{A}^t)^{-1} \, \mathbf{A}^p \, s_p$, with s^i arbitrary unit spatial vector.

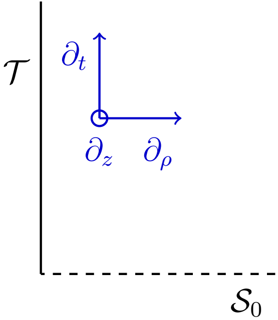
- Weak hyperbolicity: $\mathbf{P}^s$ has real eigenvalues $\forall \; s^i$
- Strong hyperbolicity: $\mathbf{P}^s$ is also diagonalizable $\forall \; s^i$

No strong hyperbolicity $\rightarrow$ **No** well-posed initial boundary value problem.



Characteristic Initial Boundary Value Problem (CIBVP)

$\longrightarrow$



Initial Boundary Value Problem (IBVP)

$$\mathbf{A}^t\,\partial_t\mathbf{u}+\mathbf{A}^p\,\partial_p\mathbf{u} \simeq 0$$

$$det(\mathbf{A}^u) = 0 \qquad\qquad det(\mathbf{A}^t) \neq 0$$

Well-posed characteristic initial value problem $\leftrightarrow$ associated well-posed Cauchy problem.

Analysis & Results

Analysis:

$$\mathbf{A}^u\,\partial_u\mathbf{u} + \mathbf{A}^r\,\partial_r\mathbf{u} + \mathbf{A}^z\,\partial_z\mathbf{u} \simeq 0$$

- Linearization on a fixed background
- 1st order reduction
- Coordinate transformation to IBVP

Result: Bondi-like PDEs are only weakly hyperbolic $\longrightarrow$ ill-posed in L_2 norm.

- Non-diagonalizable $\mathbf{A}^z$, only block diagonal
- True $\forall$ 1st order reductions

Toy models

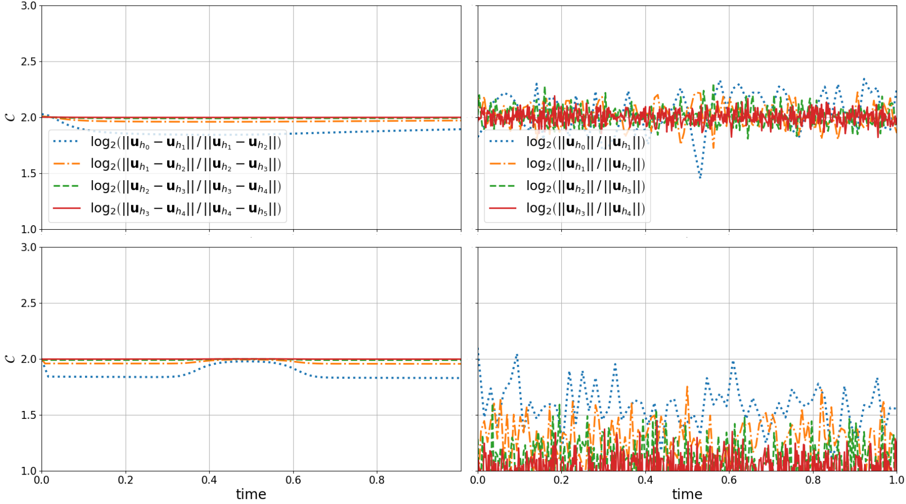
Weakly hyperbolic:

$$\begin{aligned} &\partial_x \phi = 0\,, \\ &\partial_x \psi_v - \partial_z \phi = 0\,, \\ &\partial_u \psi - F(x)\partial_x \psi - \partial_z \psi = 0\,, \end{aligned} \qquad \longrightarrow \qquad \begin{aligned} \mathbf{A}^z &= \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ \mathbf{u} &= (\phi, \psi_v,\, \psi)^T \end{aligned}$$

Strongly hyperbolic:

$$\begin{aligned} &\partial_x \phi = \psi\,, \\ &\partial_x \psi_v = \phi + \psi\,, \\ &\partial_u \psi - F(x)\partial_x \psi - \partial_z \psi = \phi\,, \end{aligned} \qquad \longrightarrow \qquad \begin{aligned} \mathbf{A}^z &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ \mathbf{u} &= (\phi, \psi_v,\, \psi)^T \end{aligned}$$

Norm convergence



Summary & Future work

Summary:

- Bondi-like formulations have far-reaching applications
  e.g. gravitational waves, strongly coupled matter
- Often provide weakly hyperbolic PDEs
  $\rightarrow$ ill-posed PDE problems **[to appear soon]**

Future work:

- Demonstrate the effect of weak hyperbolicity in a full GR code
- Find a well-posed formulation
- Use it in applications e.g. numerical holography

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Thank you!

The metrics

Bondi-Sachs axially symmetric:

$$ds^2 = \left(\frac{V}{r}e^{2\beta} - U^2 r^2 e^{2\gamma}\right) du^2 + 2e^{2\beta} du\, dr$$
$$+ 2Ur^2 e^{2\gamma} du\, d\theta - r^2 \left(e^{2\gamma} d\theta^2 + e^{-2\gamma} \sin^2\theta\, d\phi^2\right) . \qquad (1)$$

Metric functions:

$$\beta(u, r, \theta)\ , \quad \gamma(u, r, \theta)\ , \quad U(u, r, \theta)\ , \quad V(u, r, \theta)\ .$$

Ingoing Eddington-Finkelstein planary symmetric:

$$ds^2 = -Adv^2 + \Sigma^2 \left[e^B dx_\perp^2 + e^{-2B} dz^2\right] + 2drdv + 2Fdvdz\ . \qquad (2)$$

Metric functions:

$$A(v, r, z)\ , \quad B(v, r, z)\ , \quad \Sigma(v, r, z)\ , \quad F(v, r, z)\ .$$

The numerical scheme

- Implemented using **Julia 1.3.0**
- 2nd order accurate centered finite difference operators
  lopside derivatives for r_{min} and r_{max}
- Radial integration: 2-stage, 2nd order strong stability preserving method of Shu and Osher
- Time integration: 4th order Runge-Kutta
- Use of a compactified radial dimension $x = \frac{r - r_{min}}{\sqrt{c_x^2 + (r - r_{min})^2}}$
- No artificial dissipation