A (brief) Introduction to Non-perturbative Quantum Field Theory

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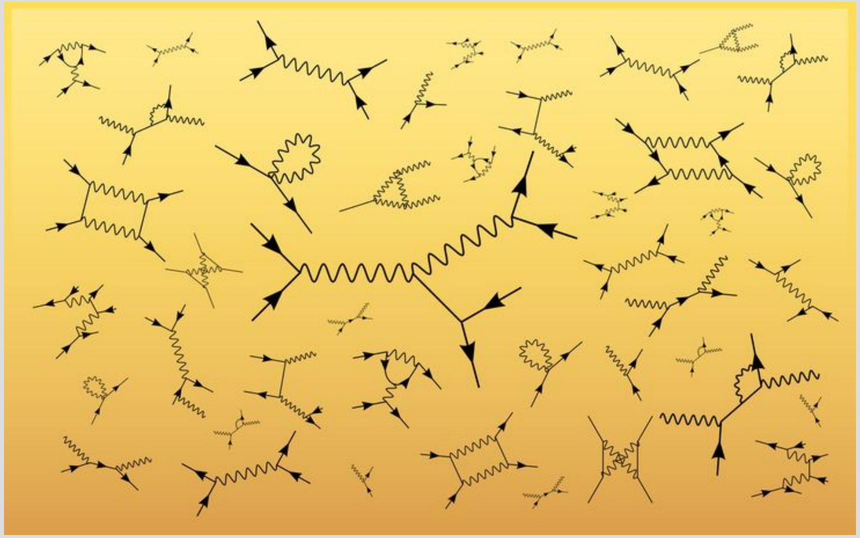
Outline

1. A non-perturbative approach. Why?
2. Local QFT in a nutshell
3. Broad implications
4. QCD and confinement
5. Summary & outlook

1. A non-perturbative approach. Why?

- Perturbation theory has proven to be an extremely successful tool for investigating problems in particle physics



But by definition this procedure is only valid in a ***weakly interacting regime***

→ Non-convergence of perturbative series
→ Observables: form factors, parton distribution functions, hadronic properties, ...
→ Confinement in QCD

- This emphasises the need for a non-perturbative approach!

→ ***Local QFT*** is one such approach

2. Local QFT in a nutshell

- Local QFT approaches are defined by a core set of axioms:

Axiom 1 (Hilbert space structure). *The states of the theory are rays in a Hilbert space $\mathcal{H}$ which possesses a continuous unitary representation $U(a,\alpha)$ of the Poincaré spinor group $\overline{\mathscr{P}_+^\uparrow}$.*

Axiom 2 (Spectral condition). *The spectrum of the energy-momentum operator P^μ is confined to the closed forward light cone $\overline{V}^+ = \{p^\mu \mid p^2 \geq 0,\, p^0 \geq 0\}$, where $U(a,1) = e^{iP^\mu a_\mu}$.*

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Axiom 5 (Relativistic covariance). *The fields $\varphi_l^{(\kappa)}(x)$ transform covariantly under the action of $\overline{\mathscr{P}_+^\uparrow}$:*

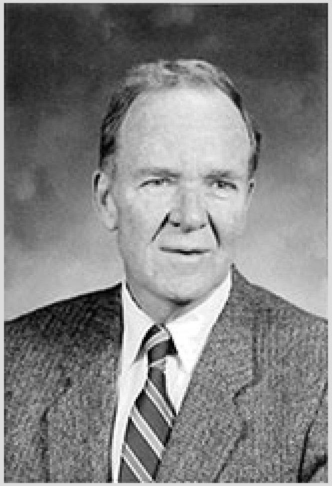
$$U(a,\alpha)\varphi_i^{(\kappa)}(x)U(a,\alpha)^{-1} = S_{ij}^{(\kappa)}(\alpha^{-1})\varphi_j^{(\kappa)}(\Lambda(\alpha)x + a)$$

where $S(\alpha)$ is a finite dimensional matrix representation of the Lorentz spinor group $\overline{\mathscr{L}_+^\uparrow}$, and $\Lambda(\alpha)$ is the Lorentz transformation corresponding to $\alpha \in \overline{\mathscr{L}_+^\uparrow}$.

Axiom 6 (Local (anti-)commutativity). *If the support of the test functions f, g of the fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$ are space-like separated, then:*

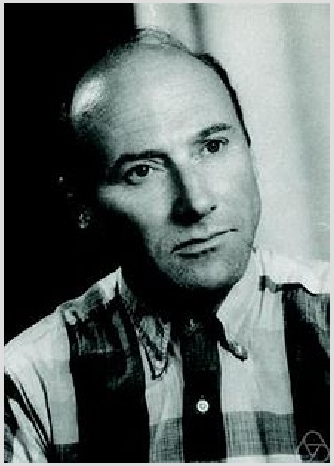
$$[\varphi_l^{(\kappa)}(f), \varphi_m^{(\kappa')}(g)]_\pm = \varphi_l^{(\kappa)}(f)\varphi_m^{(\kappa')}(g) \pm \varphi_m^{(\kappa')}(g)\varphi_l^{(\kappa)}(f) = 0$$

when applied to any state in $\mathcal{H}$, for any fields $\varphi_l^{(\kappa)}, \varphi_m^{(\kappa')}$.



A. Wightman

[R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and all that* (1964).]



R. Haag

[R. Haag, *Local Quantum Physics*, Springer-Verlag (1996).]

2. Local QFT in a nutshell

- The central idea with Local QFT is that these axioms are <u>physically motivated</u>

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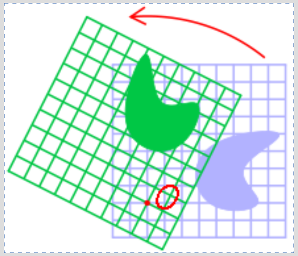
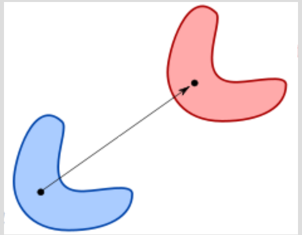
"The theory is invariant under Poincaré transformations"





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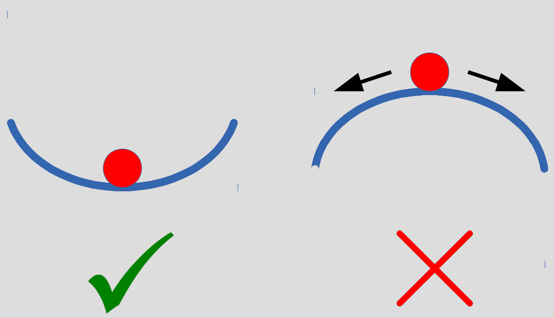
"Energy is bounded from below – the theory is stable"



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"The vacuum state is unique and looks the same to all observers"

All states are defined by acting with fields φ on the vacuum |0>

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"Quantum fields φ are operator-valued distributions"

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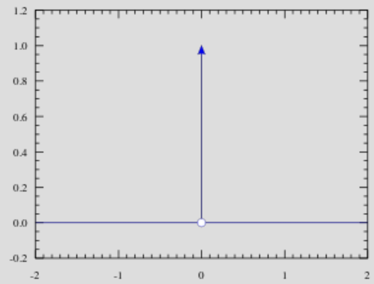
- Quantum fields $\varphi(x)$ are (operator-valued) distributions – *what difference does this make?*

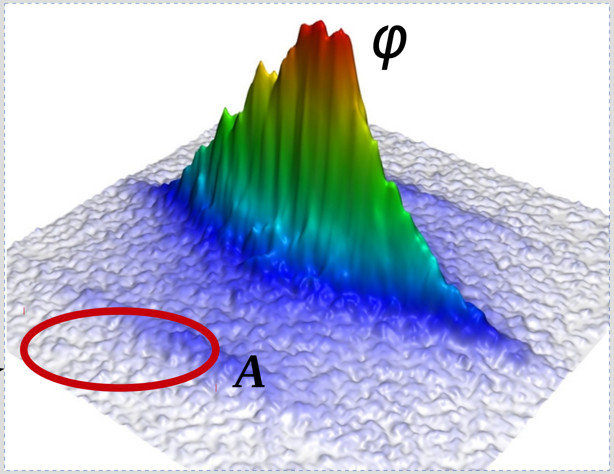


$\rightarrow$ They cannot be evaluated at a single point
e.g. Dirac delta $\delta(x)$ at $x{=}0$

$\rightarrow$ Need to 'average them out' over some spacetime region ***A***

$$\mathcal{M}_\varphi := \int_A d^4x \, \varphi(x) f(x)$$

Can think of this as the performance of a localised measurement M_φ in the region ***A*** where $f(x)$ is non-zero



[MPI Munich (2004)]

But why? – Heisenberg: $\Delta x \Delta p \sim \frac{\hbar}{2}$

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"Connects the permitted physical states with the field degrees of freedom"

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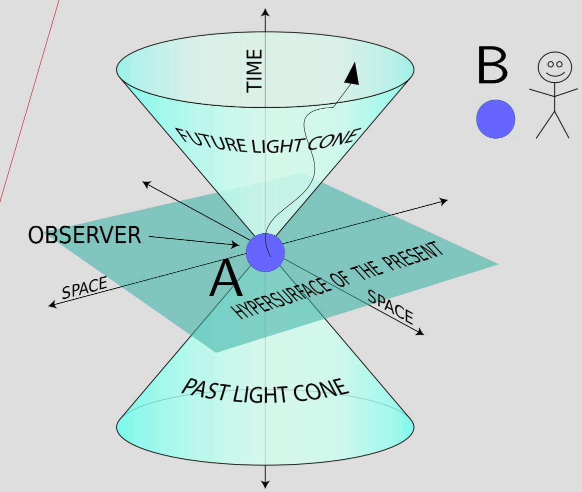
"Causality!"



A cannot send a message to B if they are space-like separated

- Despite their simplicity, these axioms give rise to many important consequences:

  → Correlation functions $\langle 0|\varphi^{(\kappa_1)}_{l_1}(x_1)\cdots\varphi^{(\kappa_n)}_{l_n}(x_n)|0\rangle$ are ***distributions***

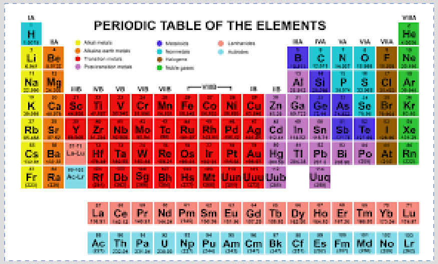
  → A QFT can be fully reconstructed from knowledge of *all* of the correlation functions

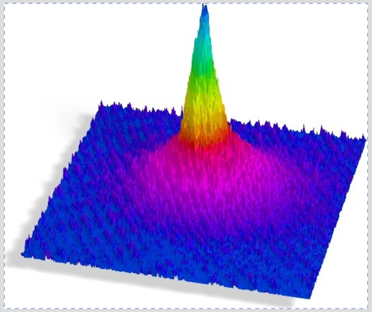
  → Connection of Minkowski and Euclidean QFTs ($t \to i\tau$)

  → *Spin-statistics Theorem*

  '*Boson*' $\leftrightarrow$ $[\varphi(x),\varphi(y)]$

  '*Fermion*' $\leftrightarrow$ $\{\psi(x),\psi(y)\}$





[The University of Toronto (2004)]

  → **CPT** is a symmetry of *any* theory

4. QCD and confinement

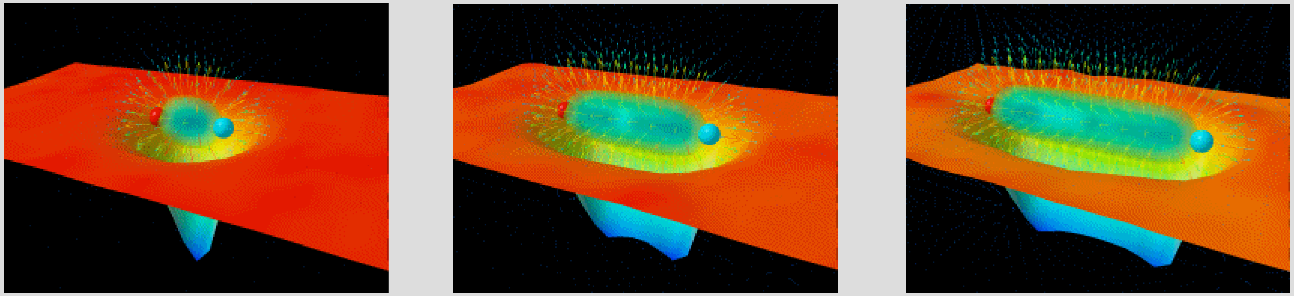
- So far, the (*Wightman*) axioms 1-6 we have discussed are known to apply to relatively simple (e.g. scalar) QFTs
- For theories of physical interest, i.e. gauge theories, significant complications arise...
- In the case of QCD, understanding these complications is essential for unravelling the <u>non-perturbative</u> structure of correlation functions

QCD correlation functions

- Enter into the calculation of bound-state observables; meson spectra, decay constants, glueball masses,...
- QCD phase diagram: effects of finite temperature and density
- *Confinement mechanism*

4. QCD and confinement

- To understand confinement is to know why nature doesn't allow the existence of physical (colour) charged states in QCD
- Or... why can't we 'pull apart' composite neutral states?



[The University of Adelaide (2015)]

- It turns out that this characteristic is closely related to the type of contributions that can appear in correlation functions
- Interestingly, contributions that prevent composite states from being pulled apart are not allowed in QFTs satisfying axioms 1-6, but <u>can</u> appear in gauge theories!

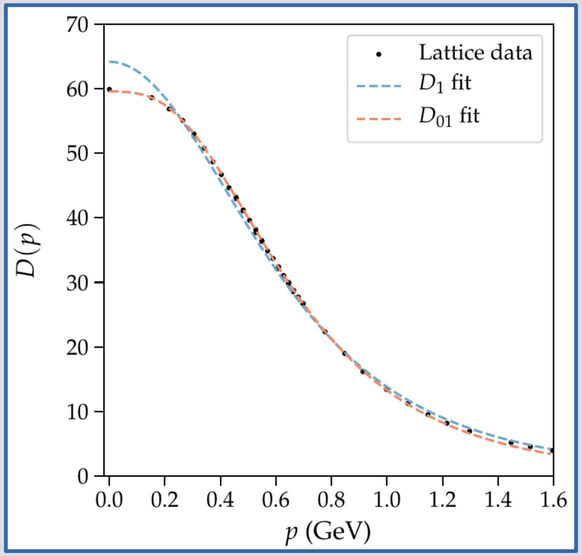
4. QCD and confinement

- An important example of these contributions are ***generalised poles***, which appear in (Euclidean) correlation functions in the form:

$$D_i(p) = \frac{Z_i}{(p^2 + m_i^2)^{i+1}}, \quad i \geq 1$$

- Using non-perturbative data, e.g. from lattice simulations, one can look for these type of contributions

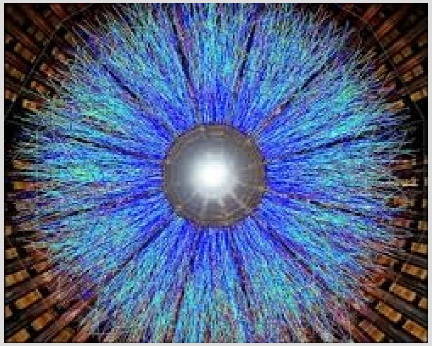
- Performing fits to (Landau gauge) lattice data for the gluon propagator **[Li, PL, Oliveira, Silva, 1907.10073]** it appears that the data *is* consistent with the existence of such poles

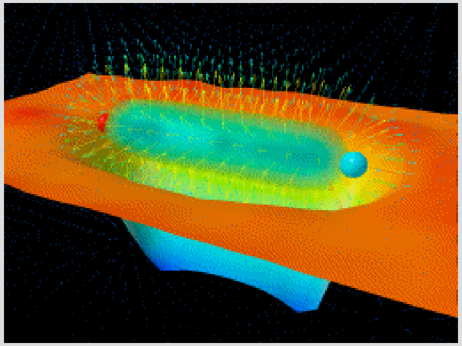


- Nevertheless, many open questions remain...

5. Summary & outlook

- Local QFT is an analytic framework that attempts to address the fundamental question: "*what is a QFT?*"
- The central idea is that a QFT is defined using a series of physically motivated axioms
- This implies general and often deep consequences – CPT symmetry, connection of Minkowski & Euclidean QFTs, Spin-statistics Theorem, ...
- In QCD, this framework can be used to provide new insights into non-perturbative phenomena such as confinement





"*Particles are the roof of the theory, not its foundation*" R. Haag

[Brookhaven National Lab]

Some further reading...

- R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and all that* (1964).
- R. Jost, *The General Theory of Quantized Fields* (1965).
- R. Haag, *Local quantum physics: Fields, particles, algebras* (1992).
- N. N. Bogolyubov, A. A. Logunov, A. I. Oksak and I. T. Todorov, *General principles of quantum field theory* (1990).
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- F. Strocchi, *An Introduction to Non-Perturbative Foundations of Quantum Field Theory* (2013).

Backup

- Things become more complicated in gauge theories...

"Local Gauss law"

$$\partial^\nu G^a_{\mu\nu} = J^a_\mu$$

(i) *Preserve positivity, lose locality*
(e.g. Coulomb gauge QED)

(ii) *Preserve locality, lose positivity*
(e.g. Landau gauge QCD)

- In option (ii), to recover locality one modifies this equation, lifting the restriction, but maintaining the constraint for **physical** states

$$\langle \text{phys}|\partial^\nu G^a_{\mu\nu} - J^a_\mu|\text{phys}\rangle = 0$$

$$\partial^\mu A^{(+)}_\mu|\text{phys}\rangle = 0$$ ***Gupta-Bleuler (QED)***

$$Q_B|\text{phys}\rangle = 0$$ ***BRST (QCD)***

- This procedure necessarily introduces both **zero** and **negative** norm (ghost) states into the theory!

$\rightarrow$ QFT axioms are modified: **Pseudo-Wightman** approach [Bogolubov et al.]