Double parton distributions in colour space.

Perturbative splitting and positivity bounds.

[arXiv:2105.08425, arXiv:2109.14304]

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Colour structure of DPDs.

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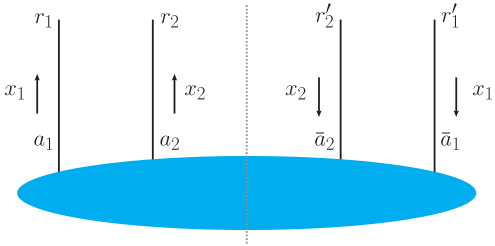
Coupling of colour indices for DPDs.

In contrast to PDFs DPDs exhibit a rich colour structure:



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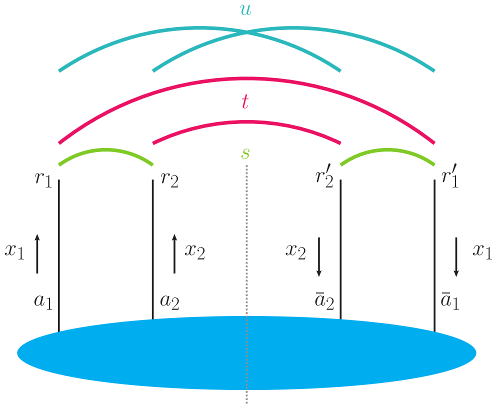

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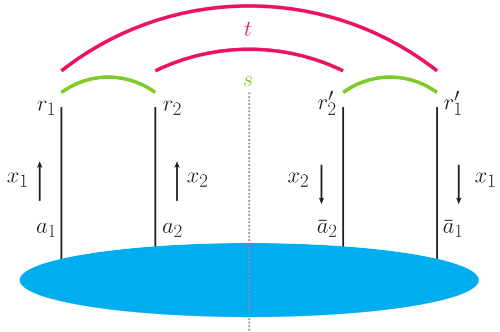

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Colour structure of DPDs.

t- and s-channel DPDs.

In the t-channel the colour indices r_i and r_i' are coupled to an irreducible representation R_i of $SU(N)$ such that $R_1 R_2$ is a singlet:

$$^{R_1R_2}F_{a_1a_2} \sim P^{r_1r_1'\,r_2r_2'}_{\overline{R}_1\overline{R}_2} F^{r_1r_1'r_2r_2'}_{a_1a_2} \qquad \text{such that} \qquad F^{r_1r_1'r_2r_2'}_{a_1a_2} \sim \sum_{R_1,R_2} P^{r_1r_1'\,r_2r_2'}_{R_1R_2}\; {}^{R_1R_2}F_{a_1a_2}$$

In the s-channel the colour indices r_1 and r_2 are coupled to an irreducible representation R and r_1' while r_2' are coupled to R' such that RR' is again a singlet:

$$F^{RR'}_{a_1a_2} \sim P^{r_1r_2\,r_1'r_2'}_{\overline{R}\,\overline{R}'} F^{r_1r_1'r_2r_2'}_{a_1a_2} \qquad \text{such that} \qquad F^{r_1r_1'r_2r_2'}_{a_1a_2} \sim \sum_{R,R'} P^{r_1r_2\,r_1'r_2'}_{RR'}\; F^{RR'}_{a_1a_2}$$

Note that the exact form of $F^{RR'}_{a_1a_2}$ and ${}^{R_1R_2}F_{a_1a_2}$ depends on the choice of normalisation.

t- and s-channel DPDs.

Depending on the parton species a_1 and a_2 one finds the following colour channels in the t- and s-channel bases:

	t-channel	s-channel
$a_1 a_2$	$R_1 R_2$	RR'
$q\bar{q}$	$11,\ 88$	$11,\ 88$
qq	$11,\ 88$	$\overline{3}3,\ 6\overline{6}$
$\overline{q}\overline{q}$	$11,\ 88$	$3\overline{3},\ \overline{6}6$
gq	$11,\ S8,\ A8$	$3\overline{3},\ \overline{6}6,\ 15\overline{15}$
$g\bar{q}$	$11,\ S8,\ A8$	$\overline{3}3,\ 6\overline{6},\ \overline{15}15$
gg	$11,\ SS,\ AA,\ SA,\ AS,\ 10\overline{10},\ \overline{10}10,\ 2727$	$11,\ SS,\ AA,\ SA,\ AS,\ 10\overline{10},\ \overline{10}10,\ 2727$

Transforming between t- and s-channel DPDs.

Using the definitions of the t- and s-channel DPDs it is straightforward to derive their transformation behaviour as:

$$F_{a_1 a_2}^{RR'} = (\boldsymbol{M}_{a_1 a_2})^{RR'}_{R_1 R_2} \, {}^{R_1 R_2}F_{a_1 a_2} \qquad \text{and} \qquad {}^{R_1 R_2}F_{a_1 a_2} = (\boldsymbol{M}_{a_1 a_2})^{R_1 R_2}_{RR'} \, F_{a_1 a_2}^{RR'}$$

where the individual entries of the transformation matrices are given by:

$$(\boldsymbol{M}_{a_1 a_2})^{RR'}_{R_1 R_2} \sim P^{r_1 r_2 \, r_1' r_2'}_{\overline{R}\,\overline{R'}} P^{r_1 r_1' \, r_2 r_2'}_{R_1 R_2} \qquad \text{and} \qquad (\boldsymbol{M}_{a_1 a_2})^{R_1 R_2}_{RR'} \sim P^{r_1 r_1' \, r_2 r_2'}_{\overline{R_1}\,\overline{R_2}} P^{r_1 r_2 \, r_1' r_2'}_{RR'}$$

Note that here the exact form of the transformation matrices depends again on the choice of normalisation for the t- and s-channel distributions.

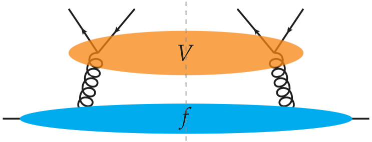
Colour non-singlet splitting DPDs at NLO.

[arXiv:2105.08425]

Perturbative splitting in DPDs.

In the limit of small distance $\boldsymbol{y}$ the leading contribution to a DPD is due to the perturbative splitting of one parton into two and can be calculated in perturbation theory:

$$^{R_1R_2}F_{a_1a_2}(x_i, y, \zeta_p, \mu) \overset{y\to 0}{=} \frac{1}{\pi y^2} \left[{}^{R_1R_2}V_{a_1a_2,a_0}(y, \zeta_p, \mu) \underset{12}{\otimes} f_{a_0}(\mu) \right] (x_i) \,,$$



where

$$\left[V \underset{12}{\otimes} f \right] (x_i) = \int_x^1 \frac{dz}{z^2} \, V\left(\frac{x_1}{z}, \frac{x_2}{z} \right) f(z)$$

LO kernels for all R_1R_2 can be found in [Diehl, Ostermeier, and Schäfer, 2012].

NLO kernels for $R_1R_2 = 11$ have been calculated in [Diehl, Gaunt, Plößl, and Schäfer, 2019].

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formally OPE of $\mathcal{O}(y,z_1)\mathcal{O}(0,z_2)$ for $y \to 0$

where

$$\left[V\underset{12}{\otimes} f\right](x_i) = \int_x^1 \frac{dz}{z^2}\, V\!\left(\frac{x_1}{z},\frac{x_2}{z}\right) f(z)$$

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Colour non-singlet splitting DPDs at NLO.

Many ingredients from the calculation of the colour singlet splitting in [Diehl, Gaunt, Plößl, and Schäfer, 2019] can be reused for the colour non-singlet case: diagrams, master integrals, Dirac algebra.

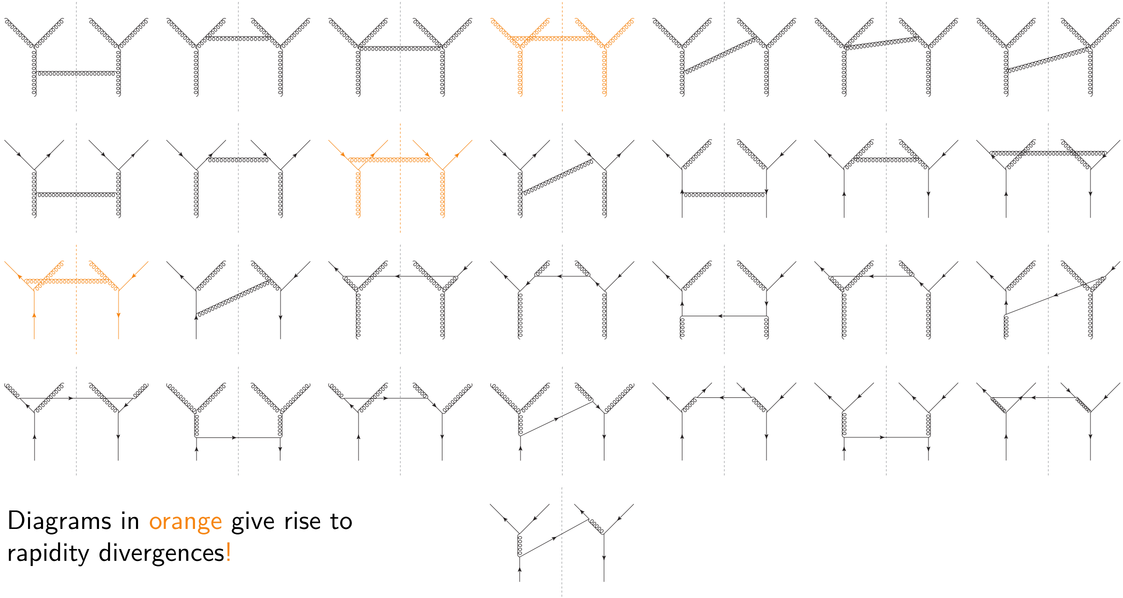
In the colour non-singlet case rapidity divergences no longer cancel after a sum over graphs and have to be treated with utmost care. For this we use two regulators:

- Collins regulator using space-like Wilson lines. [Collins, 2011]
- δ regulator. [Echevarria, Scimemi and Vladimirov, 2016]

After combining the splitting DPDs calculated with these regulators with the DPS soft factor we find matching results in both schemes!

$\longrightarrow$ First application (to our knowledge) of the Collins regulator to a two loop calculation!

Small-y limit of DPDs.



Diagrams in orange give rise to rapidity divergences!

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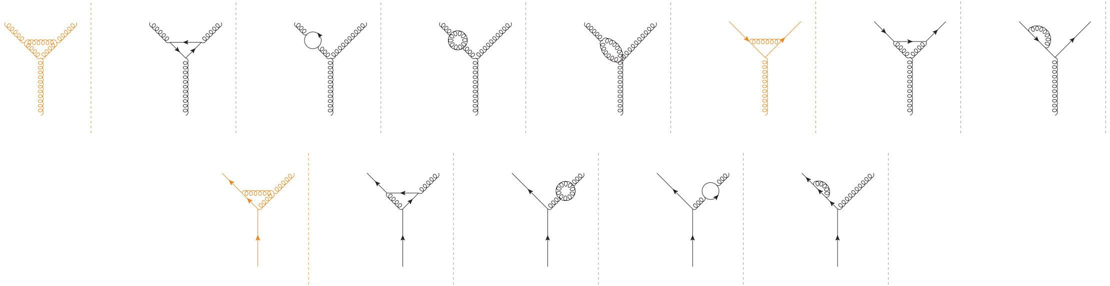


General structure of NLO colour non-singlet kernels.

Colour non-singlet kernels:

$$
\begin{aligned}
{}^{R_1R_2}V^{(2)}_{a_1a_2,a_0}(z,u,y,\mu,\zeta) &= {}^{R_1R_2}V^{[2,0]}_{a_1a_2,a_0}(z,u) + L\; {}^{R_1R_2}V^{[2,1]}_{a_1a_2,a_0}(z,u) \\
&\quad + \left(L \log\frac{\mu^2}{\zeta} - \frac{L^2}{2} + c_{\overline{\text{MS}}} \right) \frac{{}^{R_1}\gamma_J^{(0)}}{2}\; {}^{R_1R_2}V^{(1)}_{a_1a_2,a_0}(z,u)
\end{aligned}
$$

where $L = \log\frac{y^2\mu^2}{b_0^2}$ and $b_0 = 2e^{-\gamma}$ and

$$
{}^{R_1R_2}V^{[2,0]}(z,u) = {}^{R_1R_2}V^{[2,0]}_{\text{regular}}(z,u) + \delta(1-z)\; {}^{R_1R_2}V^{[2,0]}_{\delta}(u)\,,
$$

$$
{}^{R_1R_2}V^{[2,1]}(z,u) = {}^{R_1R_2}V^{[2,1]}_{\text{regular}}(z,u) + \frac{1}{[1-z]_+}\; {}^{R_1R_2}V^{[2,1]}_{+}(u) + \delta(1-z)\; {}^{R_1R_2}V^{[2,1]}_{\delta}(u)
$$

In contrast to the LO case there are – with few exceptions – no simple scaling relations between different colour channels!

Impact of NLO corrections on small y DPDs.

We study how including the NLO corrections effects the small y gg DPD for the following set of parameters:

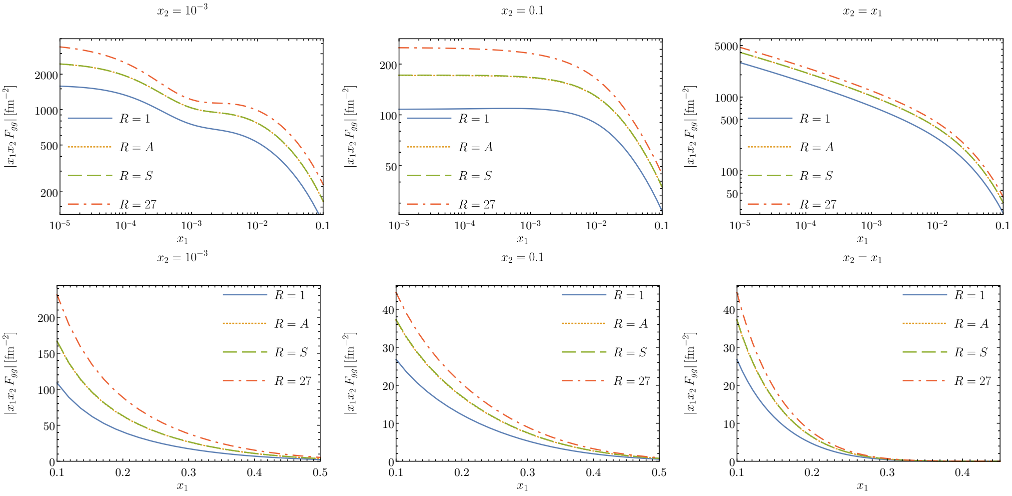
- $y = 0.022\,\text{fm}$
- $\mu = \frac{b_0}{y} = 10\,\text{GeV}$
- $x_1 x_2 \zeta_p = \mu^2 = 100\,\text{GeV}^2$

For this choice of parameters only the $V^{[2,0]}$ part of the kernels contributes to the final DPD.

In order to get a feeling for the relative importance of the logarithmic $V^{[2,1]}$ and double logarithmic $V^{(1)}$ parts we vary μ and $\sqrt{x_1 x_2 \zeta_p}$ by a factor of two around their central values.

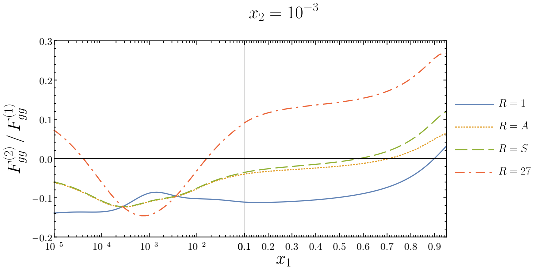
Small-y limit of DPDs.

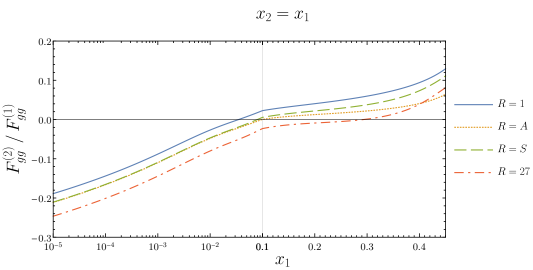
$|x_1 x_2 \, {}^{RR}F_{gg}|$.

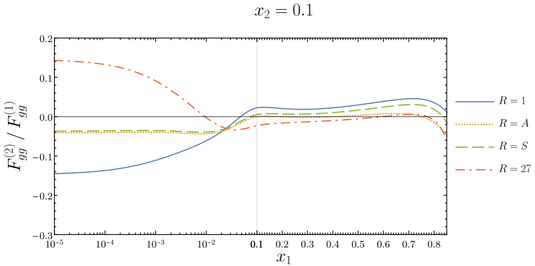


Small-y limit of DPDs.

${}^{RR}F^{(2)}_{gg} \,/\, {}^{RR}F^{(1)}_{gg}$.



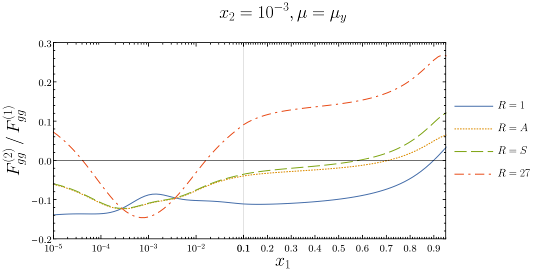


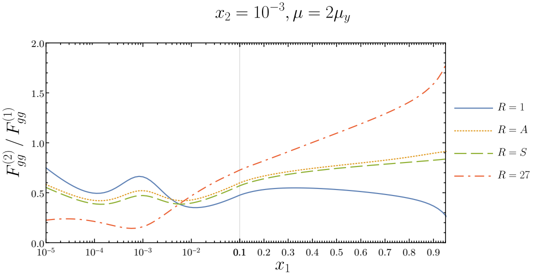


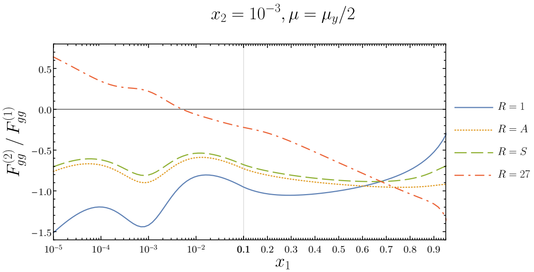
- moderate ($\mathcal{O}(10\%)$) NLO corrections.
- varied structure as a function of x_1 and x_2.
- results rather independent of PDF sets used.

Small-y limit of DPDs.

${}^{RR}F_{gg}^{(2)} \,/\, {}^{RR}F_{gg}^{(1)}$.







- large ($\mathcal{O}(100\%)$) NLO corrections for $\mu \neq \mu_y$.
- splitting form should be evaluated at $\mu \sim \mu_y$ to avoid large higher order corrections.

Violation of positivity bounds for s-channel DPDs.

[arXiv:2109.14304]

Introduction.

s-channel DPDs $F_{a_1 a_2}^{R\overline{R}}$ allow a density interpretation as the probability to find the parton pair $a_1 a_2$ in any of the $m(R)$ states of representation R. [Kasements and Mulders, 2014]

This interpretation results in the following positivity bound for s-channel DPDs:

$$F_{a_1 a_2}^{R\overline{R}} \geq 0$$

for all representations R and parton combinations $a_1 a_2$.

Check if these bounds are violated:

- Consider the small-y regime where DPDs are known from perturbation theory.
- Restrict discussion to quark-quark and quark-antiquark distributions (comparatively simple colour structure).
- Consider s-channel DPDs that vanish at LO: $F_{qq}^{\overline{3}3}$, $F_{qq}^{6\overline{6}}$, $F_{qq'}^{\overline{3}3}$, $F_{qq'}^{6\overline{6}}$, $F_{q\overline{q}}^{11}$, $F_{q\overline{q}'}^{11}$, $F_{q\overline{q}'}^{88}$.

From t- to s-channel DPDs.

The small-y splitting DPDs are given in the t-channel, s-channel DPDs are obtained via:

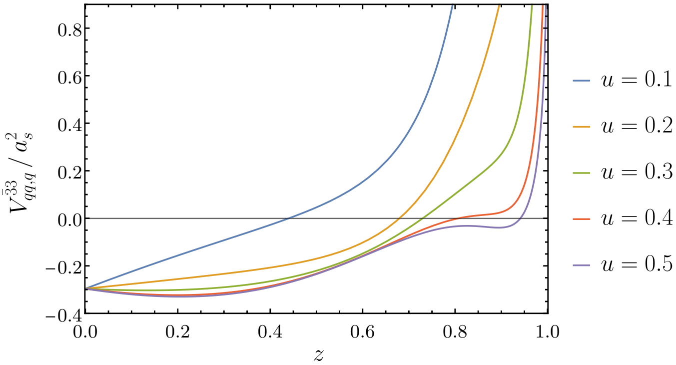
$$\begin{pmatrix} F_{qq}^{\overline{3}3} \\ F_{qq}^{6\overline{6}} \end{pmatrix} = \boldsymbol{M}_{qq} \begin{pmatrix} {}^{11}F_{qq} \\ {}^{88}F_{qq} \end{pmatrix}, \qquad \begin{pmatrix} F_{q\bar{q}}^{11} \\ F_{q\bar{q}}^{88} \end{pmatrix} = \boldsymbol{M}_{q\bar{q}} \begin{pmatrix} {}^{11}F_{q\bar{q}} \\ {}^{88}F_{q\bar{q}} \end{pmatrix}$$

for quark-quark and quark-antiquark distributions and

$$\begin{pmatrix} F_{gq}^{3\overline{3}} \\ F_{gq}^{\overline{6}6} \\ F_{gq}^{15\overline{15}} \end{pmatrix} = \boldsymbol{M}_{gq} \begin{pmatrix} {}^{11}F_{gq} \\ {}^{S8}F_{gq} \\ {}^{A8}F_{gq} \end{pmatrix}, \qquad \begin{pmatrix} F_{g\bar{q}}^{\overline{3}3} \\ F_{g\bar{q}}^{6\overline{6}} \\ F_{g\bar{q}}^{\overline{15}15} \end{pmatrix} = \boldsymbol{M}_{g\bar{q}} \begin{pmatrix} {}^{11}F_{g\bar{q}} \\ {}^{S8}F_{g\bar{q}} \\ {}^{A8}F_{g\bar{q}} \end{pmatrix}$$

for gluon-quark and gluon-antiquark distributions.

NLO splitting kernels for s-channel DPDs.



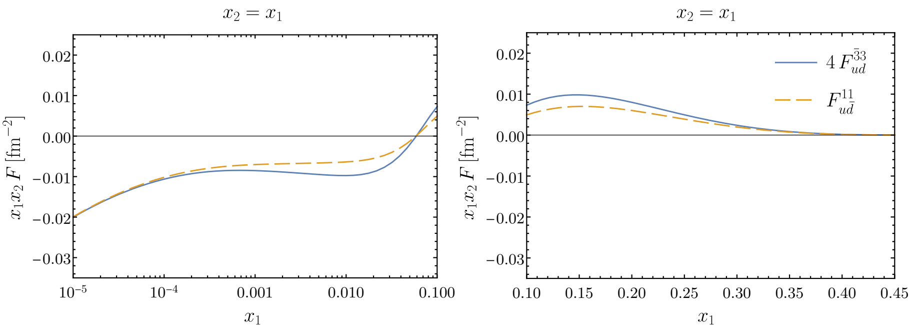
Looking at the kernels for NLO splitting DPDs in the s-channel suggests that the corresponding DPDs can become negative!

The fact that kernels computed from squared amplitudes can become negative is plausible because these kernels are defined in the $\overline{\text{MS}}$ scheme.

NLO s-channel splitting DPDs.

Using the same numerical setup as for the study of the t-channel NLO splitting contributions one can explicitly check whether the s-channel NLO splitting DPDs can violate positivity.

Consider to this end distributions with vanishing LO contributions:



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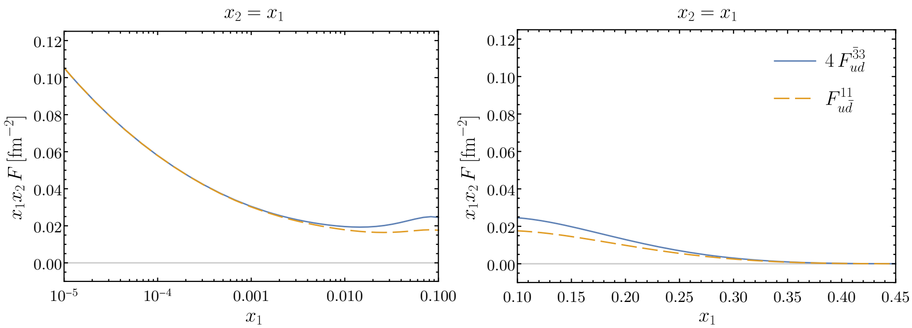

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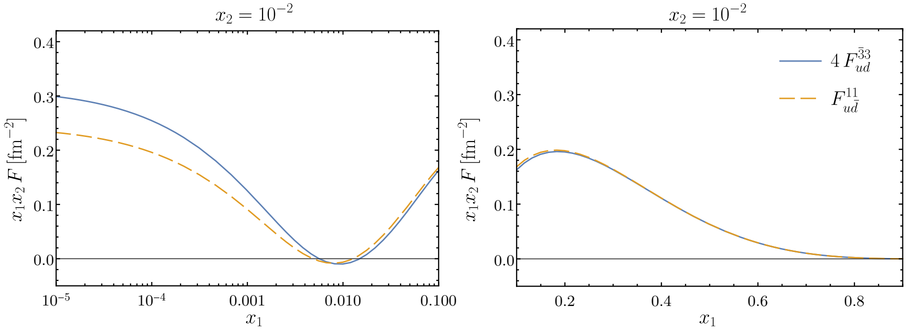

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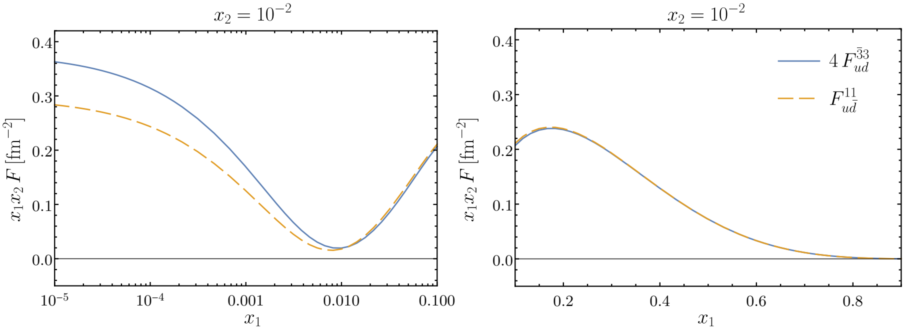

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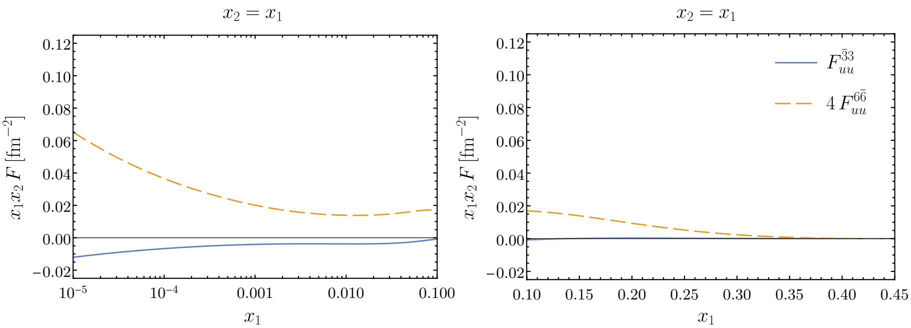

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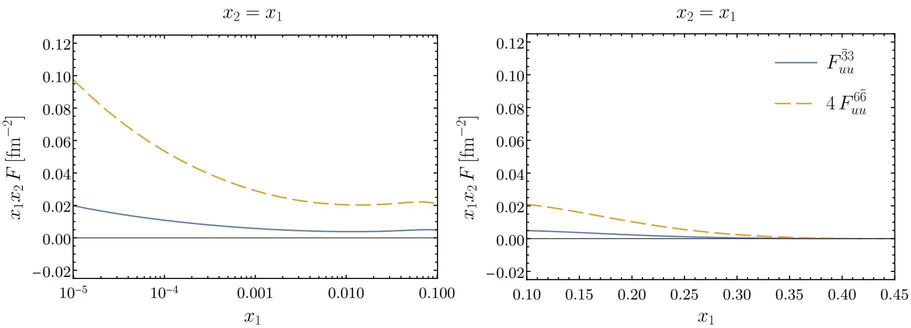

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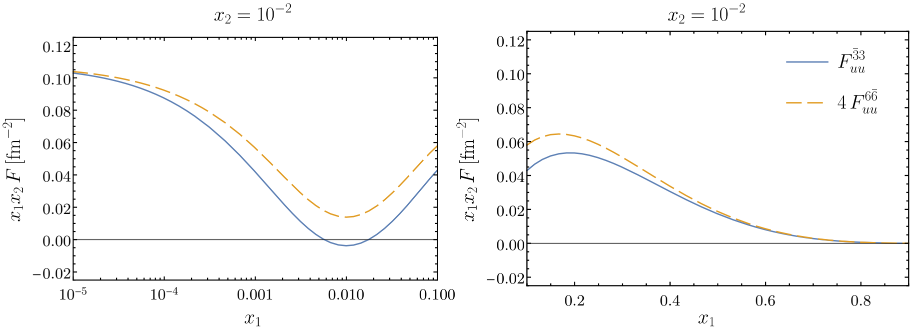

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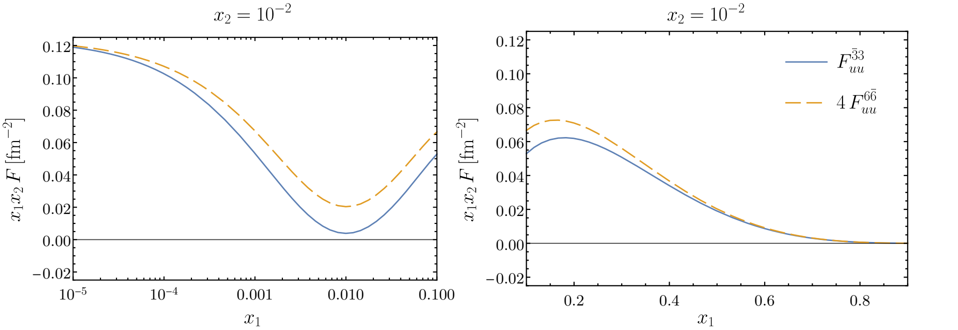

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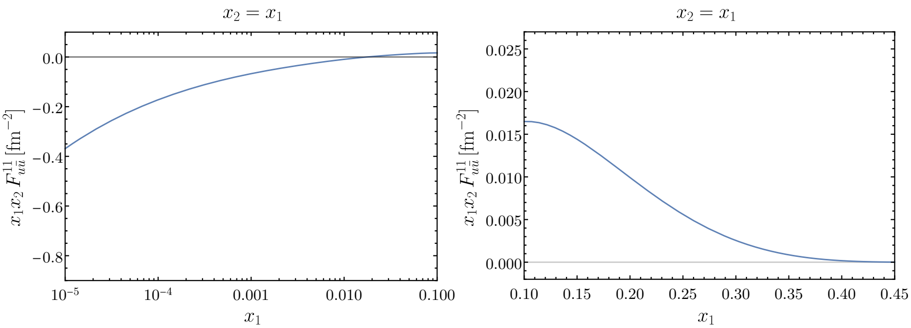

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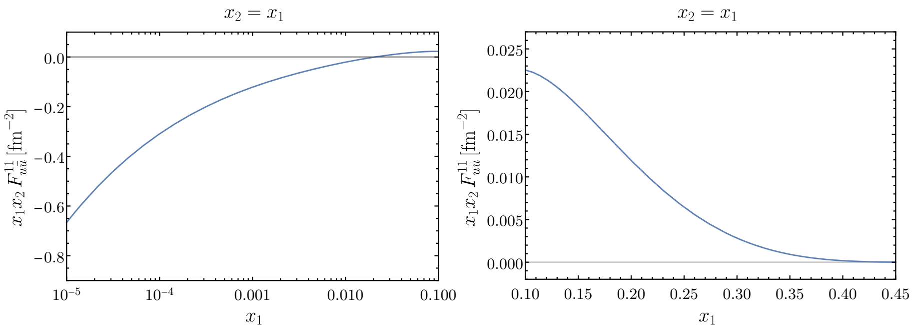

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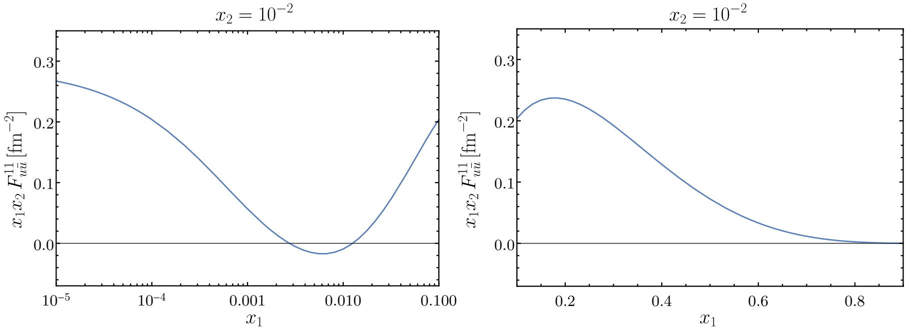

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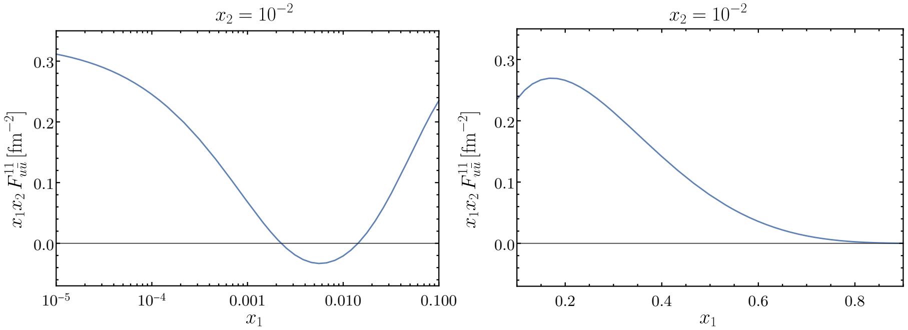


$$\mu = 1.2\mu_y$$

Evolution of s-channel DPDs.

One can show that positivity is not necessarily conserved under LO DGLAP evolution.

Consider to this end one of the small-y s-channel distributions which are zero at LO: $F^{11}_{q\bar{q}}$.

The double DGLAP equation is most naturally formulated in the t-channel, where it reads:

$$\frac{\partial}{\partial \log \mu_1^2} {}^{R_1R_2}F_{q\bar{q}}(x_i, \boldsymbol{y}, \zeta_p, \mu_i) = \left[{}^{R_1R_2}P_{qq} \underset{x_1}{\otimes} {}^{R_1R_2}F_{q\bar{q}} + \sum_{R'} {}^{R_1R'}P_{qg} \underset{x_1}{\otimes} {}^{R'R_2}F_{g\bar{q}} \right] (x_i, \boldsymbol{y}, \zeta_p, \mu_i)$$

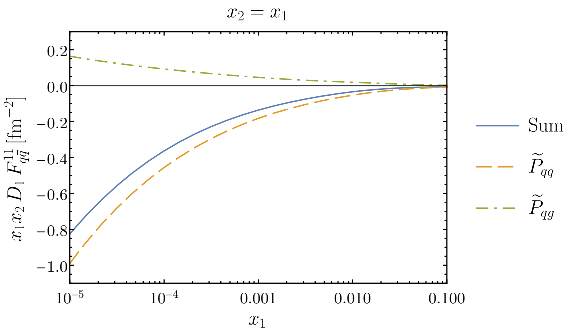
From this one can obtain the evolution equation for $F^{11}_{q\bar{q}}$ as:

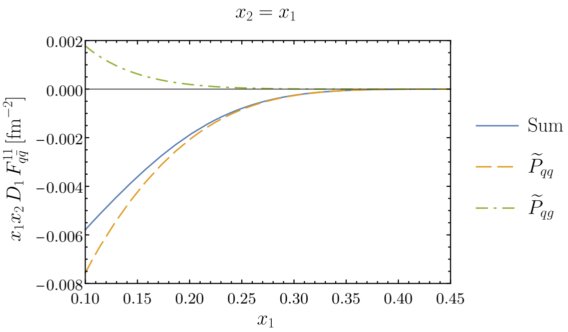
$$\frac{\partial}{\partial \log \mu_1^2} F^{11}_{q\bar{q}}(x_i, \boldsymbol{y}, \zeta_p, \mu_i) = a_s \left[\widetilde{P}_{qq} \underset{x_1}{\otimes} F^{88}_{q\bar{q}} + \frac{8}{3} \widetilde{P}_{qg} \underset{x_1}{\otimes} F^{\bar{3}3}_{g\bar{q}} - \frac{2}{9} {}^{8}\gamma_J^{(0)} \log \frac{\mu_1^2}{x_1^2 \zeta_p} F^{88}_{q\bar{q}} \right] (x_i, \boldsymbol{y}, \zeta_p, \mu_i)$$

$\longrightarrow$ If the rhs. is negative this implies that evolution to higher scales drives $F^{11}_{q\bar{q}}$ negative!

Evaluating the rhs. of the DGLAP equation.

Using the same numerical setup as before we can check explicitly if evolution to higher scales drives $F^{11}_{q\bar{q}}$ negative:





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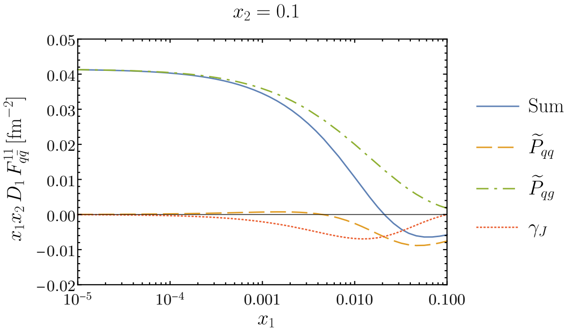

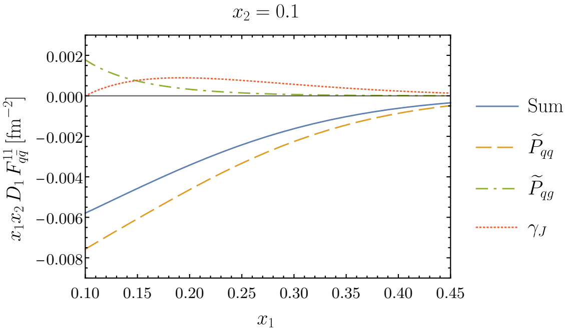

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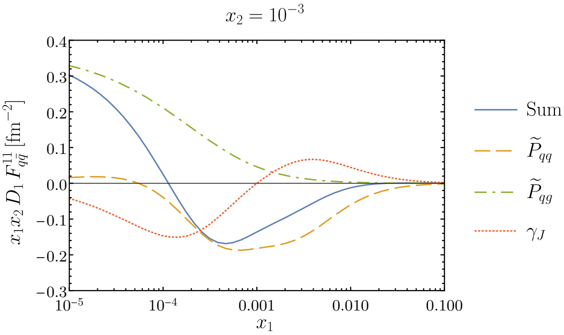

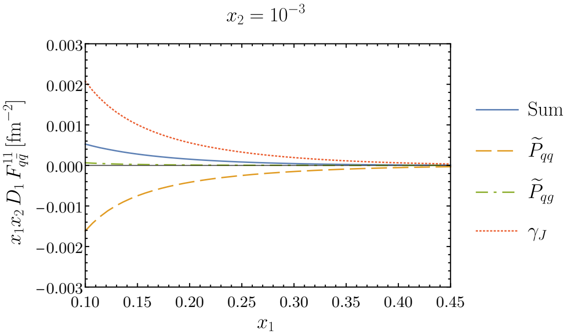

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DPDs have a non-trivial colour structure with four open colour indices coupled to a colour singlet:

- Full colour structure can be decomposed in terms of distributions projected onto definite colour representations.
- Decomposition can be made in different "bases": t-, s-, and u-channel.

In the small-y limit DPDs can be calculated perturbatively, allowing us to study colour correlations:

- Already done at LO for all R_1R_2 and NLO for $R_1R_2 = 11$.
- Now also available at NLO for all R_1R_2.

For s-channel DPDs positivity bounds can be derived, whose validity can be checked explicitly using the perturbative small-y DPDs:

- NLO DPDs can violate positivity for certain kinematics.
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Thank you for your attention!

Backup.

Rescaling of the rapidity parameter.

The rapidity parameters ζ_p and $\zeta_{\bar{p}}$ in this work are normalised as:

$$\zeta_p \zeta_{\bar{p}} = (2p^+ \bar{p}^-)^2 = s^2 \,,$$

which differs from the convention in the TMD case

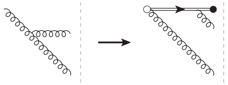
$$\zeta \bar{\zeta} = x^2 \bar{x}^2 (2p^+ \bar{p}^-)^2 = Q^4 \,,$$

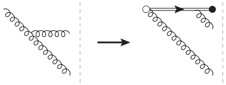
where the rapidity parameters are normalized w.r.t. the extracted parton, which would be awkward in the DPD case where parton momenta often appear in convolution integrals.

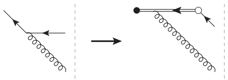
$\longrightarrow$ need to rescale the rapidity parameter in renormalisation factors and evolution kernels!

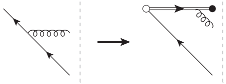
$\longrightarrow$ reason: can only depend on the plus-momentum $x_i p^+$ of the parton to which they refer!

From light-cone gauge diagrams to Wilson line diagrams in Feynman gauge.









Kinematic limits of the small y DPDs.

Large $x_1 + x_2$: Plus distributions in the kernels lead to a $\log(1 - x_1 - x_2)$ enhancement in the DPDs.

- $g \to gg$, $g \to q\bar{q}$, and $q \to qg$

Small $x_1 + x_2$: For sufficiently steep PDFs the convolution integral in the small y DPD is dominated by z^{-2} terms in the kernels (in analogy to z^{-1} terms in DGLAP kernels).

- $g \to gg$, $g \to q\bar{q}$, and $q \to gg$ (in almost all colour channels)

Small x_1 or x_2: Corresponds to the small u and small $\bar{u}$ limit, with leading contributions going as u^{-1} and $\bar{u}^{-1}$ due to slow gluons.

- $g \to gg$, $q \to gg$, $g \to qg$ (u^{-1} & $\bar{u}^{-1}$),
  $q \to qg$, and $q \to qq'$ ($\bar{u}^{-1}$)

Find two sources for this behaviour in small y DPDs:

- Explicit u^{-1} and $\bar{u}^{-1}$ terms in the kernels.
- $(1 - z\bar{u})^{-1} \sim (k^+ - k_2^+)^{-1}$, $(1 - zu)^{-1} \sim (k^+ - k_1^+)^{-1}$ and similar terms.

Colour non-singlet evolution kernels.

The colour non-singlet evolution kernels have in the t-channel double DGLAP equation for ${}^{R_1R_2}F_{q\bar{q}}$ on slide 18 have the following structure:

$$ {}^{RR'}P_{qa}(z,\zeta,\mu) = a_s \left[c_{RR'} \widetilde{P}_{qb}(z) + \delta_{RR'} \delta(1-z) \left(\frac{\gamma_q^{(0)}(\mu)}{2} + \frac{{}^{R}\gamma_J^{(0)}(\mu)}{2} \log\frac{\mu^2}{\zeta} \right) \right] + \mathcal{O}(a_s^2) $$

where

$$ c_{11} = 1\,, \qquad c_{88} = -\frac{1}{8}\,, \qquad c_{8S} = \frac{\sqrt{5}}{4}\,, \qquad c_{8A} = \frac{3}{4} $$

and

$$ \gamma_q(\mu) = 3\,C_F\,a_s(\mu) + \mathcal{O}(a_s^2)\,, \qquad {}^{1}\gamma_J(\mu) = 0\,, \qquad {}^{8}\gamma_J(\mu) = 2\,C_A\,a_s(\mu) + \mathcal{O}(a_s^2) $$