Café com Física

Spin-foam models of Quantum Space-Time

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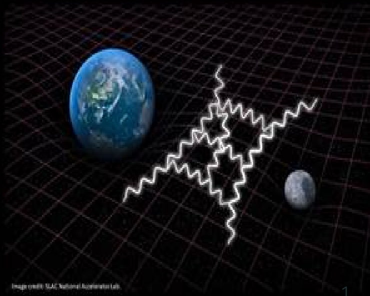

27 de novembro às 14h00

Sala de Conferências

do 3º andar

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1. Lessons from General Relativity
2. A Gauge Theory Formulation
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4. A Toy Model in 3d
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Lessons from General Relativity

Foundations of GR

- **General Principle of Relativity:** *the content of physical laws should not depend on the reference frame used to describe them.*

$$S = \int_M \mathcal{L} \tag{1}$$

- **Principle of Equivalence:** *an inertial reference frame subject to gravity is indistinguishable from an accelerated one.*

$$\mathcal{L} = \mathcal{L}(g, \varphi_{\text{matter}}) \tag{2}$$

Foundations of GR

- Einstein-Hilbert action:

$$S = \int_M \mathrm{d}^4x \sqrt{-g}\, \left[R(g) - 2\Lambda + 2\kappa\mathcal{L}_{\text{matter}}\right] \qquad (3)$$

- GR is a theory of the curvature R of a connection ∇. The connection is uniquely determined by the metric by demanding metric compatibility and vanishing torsion.

A Gauge Theory Formulation

Tetrads of spacetime

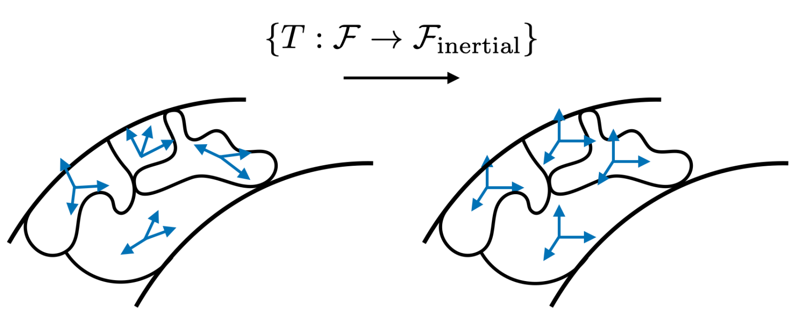
- At every point in M we may consider a set of 4 linearly independent vectors $\{e_I\}$, $e_I \in TM$ such that $g(e_I, e_J) = \eta_{IJ}$. Can be thought of as maps

$$
\begin{aligned}
e : M \times \mathbb{R}^{3,1} &\to TM \\
(x, \hat{e}_I) &\mapsto (x, e_I^\mu \partial_\mu)
\end{aligned}
\tag{4}
$$

- The metric at every point can be reconstructed from the duals $\theta^I(e_J) = \delta^I_J$ with $g = \eta_{IJ}\theta^I \otimes \theta^J$.
- Tetrads are defined up to the the isometry group $G = SO(3,1)$ of the Minkowski metric, $\Lambda\eta\Lambda^T = \eta$.

Tetrads of spacetime

- General Relativity can be reformulated in terms of these tetrads.



- But there is a gauge redundancy $G = SO(3,1)$!

Tetrads of spacetime

There is a gauge theory for tetrads:

- Gauge encoded in a principal bundle $P(G, M) \xrightarrow{\pi} M$ and connection form $\omega \in \Omega(P, \mathfrak{so}(3,1))$.
- Vector bundle $E = P \times_\rho \mathbb{R}^{3,1}$ with fundamental representation ρ.
- E and TM are both vector bundles of the same finite dimension $\Rightarrow$ we have an isomorphism acting on canonical sections of E,

$$\begin{aligned} e : E &\to TM \\ \sigma_I &\mapsto e_I{}^\mu \partial_\mu \,. \end{aligned} \tag{5}$$

We recover the tetrads from $e^I = e(\sigma^I)$, and hence a metric $g = \eta_{IJ}\theta^I \otimes \theta^J$.

Tetradic Palatini theory

The connection in P induces one in TM, from which R can be derived. One can show the EH action becomes

$$S_T = \int_M F^{IJ}[A] \wedge \star(\theta_I \wedge \theta_J) - \frac{\Lambda}{6}\, \theta^I \wedge \theta^J \wedge \star(\theta_I \wedge \theta_J)\,. \tag{6}$$

- The metric g from θ^I turns out to naturally allow metric compatibility.
- Varying the action wrt to the gauge field A imposes vanishing curvature.

This is the same curvature from GR!

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Other formulations

We can play around with this action:

- Holst action adds irrelevant term,

$$S_H = \int_M F^{IJ}[A] \wedge \left(\star + \frac{1}{\gamma}\right) (\theta_I \wedge \theta_J) \tag{8}$$

- Can extract tetrads from a two-form B using constraints,

$$S_{cBF} = \int_M B_{IJ} \wedge F^{IJ}[A] + \varphi_{IJKL} B^{IJ} \wedge B^{KL} \,, \tag{9}$$

$$\frac{\delta}{\delta\varphi} \to B^{IJ} \wedge B^{KL} = \varepsilon^{IJKL} V \,. \tag{10}$$

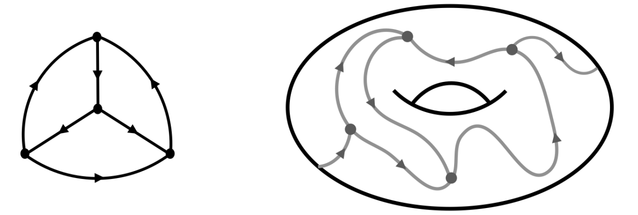
States and Observables

Spin-networks

We consider a compact-G gauge theory over an embedded graph φ.

- Space of connections $\mathcal{A}_\varphi = G^{|\mathcal{E}_\varphi|}$, $g_\gamma = \mathcal{P}\exp\left\{-\int_\gamma A\right\}$
- Gauge group $\mathcal{G}_\varphi = G^{|\mathcal{V}_\varphi|}$, $g_\gamma \to g_t^{-1} g_\gamma g_s$

Quantum states will be elements of $\mathcal{H} = L^2(\mathcal{A}_\varphi/\mathcal{G}_\varphi)$.



Spin-networks

- The action of $\mathcal{G}_\varphi$ on $\mathcal{A}_\varphi$ induces a representation of G on $L^2(G)$ "from both sides" and "at each edge". Peter-Weyl theorem:

$$L^2(\mathcal{A}_\varphi) \simeq \bigotimes_{e\in\mathcal{E}_\varphi} \bigoplus_{\lambda\in\Lambda} \mathcal{H}^\lambda \otimes \mathcal{H}^{*\lambda} \tag{11}$$

- This can be massaged to

$$L^2(\mathcal{A}_\varphi) \simeq \bigoplus_{\Lambda\to\mathcal{E}_\varphi} \bigotimes_{v\in\mathcal{V}_\varphi} \left(\bigotimes_{e\in\mathcal{S}_v} \mathcal{H}_e \otimes \bigotimes_{e\in\mathcal{T}_v} \mathcal{H}_e^* \right), \tag{12}$$

- The Hilbert space is therefore

$$\begin{aligned} L^2(\mathcal{A}_\varphi/\mathcal{G}_\varphi) &\simeq \bigoplus_{\Lambda\to\mathcal{E}_\varphi} \bigotimes_{v\in\mathcal{V}_\varphi} \mathrm{Inv} \left(\bigotimes_{e\in\mathcal{S}_v} \mathcal{H}_e \bigotimes_{e\in\mathcal{T}_v} \mathcal{H}_e^* \right) \\ &\simeq \bigoplus_{\Lambda\to\mathcal{E}_\varphi} \bigotimes_{v\in\mathcal{V}_\varphi} \mathrm{Int} \left(\bigotimes_{e\in\mathcal{S}_v} \mathcal{H}_e, \bigotimes_{e\in\mathcal{T}_v} \mathcal{H}_e \right) \end{aligned} \tag{13}$$

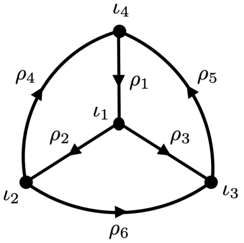
Spin-networks

- A general state in $\mathcal{H}$ will have the form

$$|\psi\rangle = \bigoplus_{\Lambda\to\mathcal{E}_\varphi} \bigotimes_{v\in\mathcal{V}_\varphi} (c_v)^{i_1\dots i_m}_{j_1\dots j_n} (\iota_v)^{j_1\dots j_n}_{i_1\dots i_m} \tag{14}$$

- Wave-functions of the connection are constructed as

$$\psi(A) = \left(\rho^{v_1}_{\text{out}}(H_A^{e^{\text{out}}_{v_1}}) \circ \iota^{v_1} \circ \rho^{v_1}_{\text{in}}(H_A^{e^{\text{in}}_{v_1}})\right) \circ \dots \circ \left(\rho^{v_n}_{\text{out}}((H_A^{e^{\text{out}}_{v_n}})) \circ \iota^{v_n} \circ \rho^{v_n}_{\text{in}}(H_A^{e^{\text{in}}_{v_n}})\right)$$



$$\psi(A) = (\rho_1)^i_a(\iota_1)^a_{bc}(\rho_2)^b_j(\rho_3)^c_d(\rho_6)^k_e(\iota_3)^{de}_f(\rho_5)^f_h(\rho_4)^l_g(\iota_4)^{gh}_i(\iota_2)^j_{kl}$$

Observables

For concreteness, we choose $G = SU(2)$ and focus on

$$\mathcal{H} = \bigoplus_{j_0, j_1, j_2, j_3} \mathrm{Inv}_{SU(2)} \left(j_0 \otimes j_1 \otimes j_2 \otimes j_3 \right) . \tag{15}$$

- Define the operators $B_0^i = J^i \otimes 1 \otimes 1 \otimes 1$, $B_1^i = 1 \otimes J^i \otimes 1 \otimes 1$, etc.
- The B_μ^i generate the action of G on $\mathcal{T} = \bigotimes_i j_i$, so

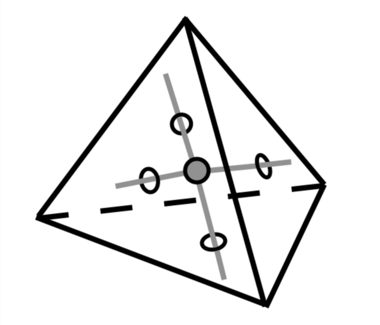
$$\mathcal{H} \simeq \left\{ |\psi\rangle \in \mathcal{T} \,\middle|\, \sum_\mu B_\mu |\psi\rangle = 0 \right\} . \tag{16}$$

- Area and volume operators:

$$\begin{aligned} A_\mu &= \sqrt{B_\mu \cdot B_\mu} \\ V &= \sqrt{\frac{1}{3!} |\varepsilon_{ijk} B_2^i B_1^j B_3^k|} \, . \end{aligned} \tag{17}$$

For $SU(2)$ the area operator is just the Casimir,

$$A_\mu |\psi\rangle = \sqrt{j_\mu(j_\mu + 1)}\, |\psi\rangle \tag{18}$$



A Toy Model in 3d

3d BF theory

We start with the Riemannian Palatini theory in 3d, setting $G = SO(3)$,

$$S_T^{(3)} = \int_M \varepsilon_{IJK} F^{IJ}[A] \wedge \theta^K \,. \tag{19}$$

It can be written as a *BF* theory,

$$S_{BF} = \int_M \mathrm{Tr}(F[A] \wedge B) \,, \tag{20}$$

due to the natural identification $\mathfrak{so}(3) \simeq T^*M$.

Fixing $M = \mathbb{R} \times M'$ and the gauge $A_0 = 0$,

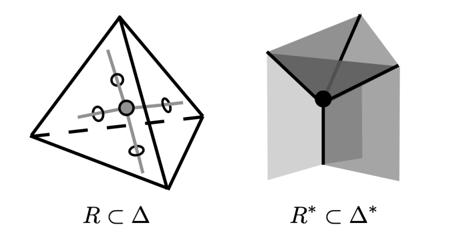
$$\frac{\partial \mathcal{L}}{\partial \dot{A}} = B \,. \tag{21}$$

Path integral quantization

- Formal path integral:

$$Z(M) = \int \mathcal{D}A\mathcal{D}B\, e^{i\int_M \mathrm{Tr}(F[A]\wedge B)}$$
$$= \int \mathcal{D}A\, \delta(F[A])\,.$$

- Discretization via triangulation Δ and dual 2-complex Δ^*:



$$Z(\Delta^*) = \int \prod_{e\in\mathcal{E}} \mathrm{d}g_e \prod_{f\in\mathcal{F}} \delta\left(\prod_{e\in\partial f} g_e\right)\,, \tag{22}$$

The Lie group delta is a function on the group, and can be expanded by Peter-Weyl as

$$\delta(g) = \sum_{\lambda \in \Lambda} \dim(\rho_\lambda)\chi_{\rho_\lambda}(g)\,. \tag{23}$$

$$\begin{aligned}
Z(\Delta^*) &= \int \prod_{e \in \mathcal{E}} \mathrm{d}g_e \prod_{f \in \mathcal{F}} \left(\sum_{\lambda \in \Lambda} \dim(\rho_\lambda) \mathrm{Tr}\left[\prod_{e \in \partial f} \rho_\lambda(g_e) \right] \right) \\
&= \sum_{\Lambda \to \mathcal{F}} \int \prod_{e \in \mathcal{E}} \mathrm{d}g_e \prod_{f \in \mathcal{F}} \left(\dim(\rho_f) \mathrm{Tr}\left[\prod_{e \in \partial f} \rho_f(g_e) \right] \right) \\
&= \sum_{\Lambda \to \mathcal{F}} \left[\prod_{f \in \mathcal{F}} \dim(\rho_f) \right] \int \prod_{e \in \mathcal{E}} \mathrm{d}g_e \, \mathrm{Tr}_{f \in \mathcal{F}} \left[\prod_{f \in \mathcal{F}} \left(\prod_{e \in \partial f} \rho_f(g_e) \right) \right] \\
&= \sum_{\Lambda \to \mathcal{F}} \left[\prod_{f \in \mathcal{F}} \dim(\rho_f) \right] \mathrm{Tr}_{f \in \mathcal{F}} \left[\prod_{e \in \mathcal{E}} \left(\int \mathrm{d}g_e \prod_{f :\, e \in \partial f} \rho_f(g_e) \right) \right]
\end{aligned}$$

Path integral quantization

There is a projector onto the invariant subspace for each edge,

$$\pi_e : \bigotimes_{f\in\mathcal{S}(e)} \mathcal{H}_{\rho_f} \bigotimes_{f\in\mathcal{T}(e)} \mathcal{H}^*_{\rho_f} \to \mathrm{Inv}\left(\bigotimes_{f\in\mathcal{S}(e)} \mathcal{H}_{\rho_f} \bigotimes_{f\in\mathcal{T}(e)} \mathcal{H}^*_{\rho_f}\right)$$
$$\pi_e = \int_{SU(2)} \mathrm{d}g_e \bigotimes_{f\in\mathcal{S}(e)} \rho_f(g_e) \bigotimes_{f\in\mathcal{T}(e)} \rho^*_f(g_e)\,, \tag{24}$$

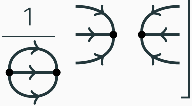
In 3d each edge is shared by 3 faces, so the projector is simply

$$\pi_e = \int_{SU(2)} \mathrm{d}g_e\; \rho_1(g)\rho_2(g)\rho_3(g)\,. \tag{25}$$

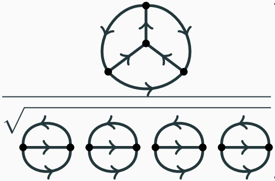
Path integral quantization

$$Z(\Delta^*) = \sum_{\Lambda\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} \dim(\rho_f)\right] \mathrm{Tr}_{f\in\mathcal{F}} \left[\prod_{e\in\mathcal{E}} \pi_e\right]$$

$$= \sum_{\Lambda\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} \dim(\rho_f)\right] \mathrm{Tr}_{f\in\mathcal{F}} \left[\prod_{e\in\mathcal{E}} \cdots \right]$$

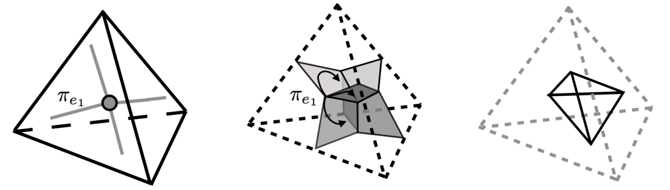


$$= \sum_{\Lambda\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} \dim(\rho_f)\right] \left[\prod_{v\in\mathcal{V}} \cdots \right] \qquad (26)$$



Boundary states

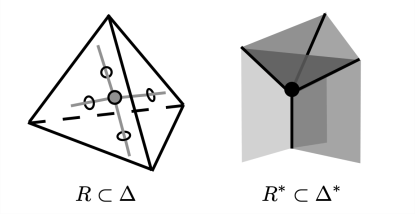
A Spinfoam $F(\Delta^*, \rho, \iota)$ is a 2-complex colored with algebraic data.

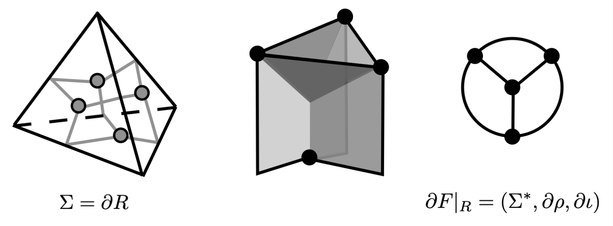


It induces spin-network boundary states, with an amplitude map given by

$$A(\partial F|_R) = \left[\prod_{f \in (\mathcal{F} \cap R^*)} \dim(\rho_f) \right] \left[\prod_{v \in (\mathcal{V} \cap R^*)} 6j \right]. \tag{27}$$

Boundary states





EPRL Model

4d BF with constraints

Tetradic Palatini from BF theory, with $G = SL(2,\mathbb{C})$:

$$S = \int_M F^{IJ} \wedge \left(1 + \frac{1}{\gamma}\star\right) B_{IJ}\,, \quad B^{IJ} = \pm\star(\theta^I \wedge \theta^J) \tag{28}$$

Construct smeared fields $b_f^{IJ} = \int_{f\subset\Delta} B^{IJ}$. The bivector $\star b_f$ is simple iff

$$n_I(\star b_f)^{IJ} = 0, \quad \text{for all } f \text{ in the same tetrahedron } t \tag{29}$$

Variable conjugated to A is associated to $\tilde{b}_f^{IJ} = \left(1 + \frac{1}{\gamma}\star\right) b_f^{IJ}$, so quantization will relate it to generators J^{IJ},

$$b_f^{IJ} = \frac{\gamma^2}{\gamma^2+1}\left(1 - \frac{1}{\gamma}\star\right) J_f^{IJ}\,. \tag{30}$$

We will fix $n^I = \delta^I_0$, corresponding to setting all tetrahedra to be spacelike.

4d BF with constraints

One then finds that the geometricity constraints restrict the representations of $SL(2,\mathbb{C})$

$$|(p,n);j,m\rangle \overset{\text{constr.}}{\to} |(2\gamma j,2j);j,m\rangle \ , \tag{31}$$

making the Hilbert spaces into $SU(2)$ ones. There is an isomorphism

$$\begin{gathered} I : L^2(SL(2,\mathbb{C}))|_{\text{constr.}} \to L^2(SU(2)) \\ |(2\gamma j,n);2j,m\rangle \mapsto |j,m\rangle \ , \end{gathered} \tag{32}$$

so the boundary space is

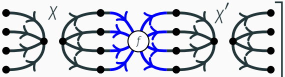
$$H_\Sigma = \bigoplus_{j\to\mathcal{F}} \bigotimes_{e\in\mathcal{E}} \mathrm{Inv}_{SU(2)} \left(\bigotimes_{f\in\mathcal{S}(e)} \mathcal{H}_{j_f}^{(2j_f\gamma,2j_f)} \bigotimes_{f\in\mathcal{T}(e)} \mathcal{H}_{j_f}^{*(2j_f\gamma,2j_f)} \right) . \tag{33}$$

Path integral quantization

$$
\begin{aligned}
Z(\Delta^*) &= \int \prod_{e\in\mathcal{E}} \mathrm{d}g_e \prod_{f\in\mathcal{F}} \left(\sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \mathrm{d}p \; (n^2+p^2) \mathrm{Tr}\left[\prod_{e\in\partial f} (D^*)^{\chi}\,(g_e) \right] \right) \\
&= \sum\!\!\!\!\!\!\!\int_{\chi\to\mathcal{F}} \int \prod_{e\in\mathcal{E}} \mathrm{d}g_e \prod_{f\in\mathcal{F}} \left((n^2+p^2)_f \mathrm{Tr}\left[\prod_{e\in\partial f} D^*_f\,(g_e) \right] \right) \\
&= \sum\!\!\!\!\!\!\!\int_{\chi\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} (n^2+p^2)_f \right] \int \prod_{e\in\mathcal{E}} \mathrm{d}g_e \mathrm{Tr}_{f\in\mathcal{F}} \left[\prod_{f\in\mathcal{F}} \left(\prod_{e\in\partial f} D^*_f\,(g_e) \right) \right] \\
&= \sum\!\!\!\!\!\!\!\int_{\chi\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} (n^2+p^2)_f \right] \mathrm{Tr}_{f\in\mathcal{F}} \left[\prod_{e\in\mathcal{E}} \left(\int \mathrm{d}g_e \prod_{f\colon e\in\partial f} D^*_f\,(g_e) \right) \right] ,
\end{aligned}
$$

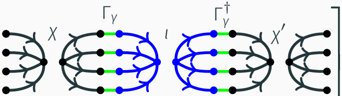
$$\not\sum_{\chi\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} (n^2+p^2)_f\right] \mathrm{Tr}_{f\in\mathcal{F}} \left[\prod_{e\in\mathcal{E}} \pi_e \circ f \circ \pi_e\right]$$

$$= \sum_{j\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} j_f^2(\gamma^2+1)\right] \mathrm{Tr}_{f\in\mathcal{F}} \left[\prod_{e\in\mathcal{E}} \int \mathrm{d}\chi\,\mathrm{d}\chi'\ \Delta_\chi \Delta_{\chi'}\right.$$



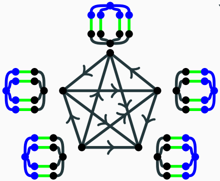
$$\left.\vphantom{\prod}\right]$$

$$= \sum_{j\to\mathcal{F}} [\ldots]\,\mathrm{Tr}_{f\in\mathcal{F}} \left[\prod_{e\in\mathcal{E}} \sum_{\iota} \int \mathrm{d}\chi\,\mathrm{d}\chi'\ \Delta_\chi \Delta_{\chi'}\right.$$



$$\left.\vphantom{\prod}\right]$$

$$= \sum_{j\to\mathcal{F}} \left[\prod_{f\in\mathcal{F}} j_f^2(\gamma^2+1)\right] \left[\prod_{v\in\mathcal{V}} \left(\prod_{i=1}^{5} \sum_{\iota_i} \int \mathrm{d}\chi_i \Delta_{\chi_i}\right.\right.$$



$$\left.\left.\vphantom{\prod}\right)\right]$$