Problems for the Multi-Higgs session

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1 Gauge Theories

1.1 Show that the QED Lagrangian,

$$\begin{aligned}\mathcal{L}_{\text{QED}} =& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{D} - m)\psi \\ \equiv& \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}\end{aligned}$$

where ($Q_e = -1$ for the electron)

$$\begin{aligned}D_\mu \equiv& \partial_\mu + ieQ_eA_\mu \\ \mathcal{L}_{\text{free}} \equiv& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\not{\partial} - m)\psi \\ \mathcal{L}_{\text{int}} \equiv& -eQ_e\overline{\psi}\gamma_\mu\psi A^\mu\end{aligned}$$

is invariant under the finite, spacetime dependent transformations,

$$\psi' = e^{i\alpha(x)}\psi, \quad A'_\mu = A_\mu - \frac{1}{e\,Q_e}\partial_\mu\alpha(x)$$

1.2 In this problem we want to show that the Lagrangian for a non-abelian (Yang-Mills) theory

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$$

where

$$F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\ f^{bca}A^b_\mu A^c_\nu$$

and f^{abc} are the completely antisymmetric group structure constants, is invariant under the infinitesimal transformations,

$$\delta A^a_\mu = -f^{bca}\ \varepsilon^b\ A^c_\mu - \frac{1}{g}\ \partial_\mu\varepsilon^a\ .$$

a) First show that

$$\delta F^a_{\mu\nu} = -f^{bca}\epsilon^b F^c_{\mu\nu}$$

To do this you need to use the Jacobi identity for the structure constants

$$f^{bdc}f^{ace} + f^{dac}f^{bce} + f^{abc}f^{dce} = 0$$

b) Use the previous result to show the invariance of the Lagrangian

c) Comment on the differences with respect to QED.

1.3 A mass term for a gauge field A^a_μ would have the form,

$$\mathcal{L}_{\text{massa}} = \frac{1}{2}\, m^2 A^a_\mu A^{a\mu} \ .$$

Show that this term is not gauge invariant. For this consider the infinitesimal transformations in Problem 1.2 and evaluate

$$\delta\mathcal{L}_{\text{massa}}$$

Notice that this happens for non-abelian as well as abelian theories.

1.4 Consider the following Lagrangian

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi$$

where ϕ is a complex scalar field.

a) Show that the equations of motion lead to the Klein-Gordon equations

b) Show that the Lagrangian is invariant under the transformations

$$\phi' = e^{-ie\alpha}\,\phi \qquad ; \qquad \alpha = \text{constant}$$

c) Show that if the action

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu\phi)$$

is invariant under the transformation

$$\phi'_i = \phi_i - i\varepsilon\lambda_{ij}\ \phi_j$$

where ε is infinitesimal and λ_{ij} are constants then there exists a conserved current, that is

$$\partial_\mu J^\mu = 0$$

where

$$J^\mu = -i\lambda_{ij}\ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\ \phi_j$$

This result is known as the Noether theorem for internal symmetries.

d) Apply this result to the Lagrangian above.

e) Show that if $\alpha = \alpha(x)$ then the Lagrangian

$$\mathcal{L} = (\partial_\mu + ieA_\mu)^* \phi^* (\partial_\mu - ieA_\mu) \phi - m^2\phi^*\phi$$

is invariant under the transformations

$$\phi' = e^{-ie\alpha(x)} \phi$$

if A_μ transforms in an appropriate way. Find this transformation law and comment on the result.

2 Goldstone Theorem and Higgs Mechanism

2.1 Consider a theory defined by the following Lagrangian,

$$\mathcal{L} = \frac{1}{2} \partial_\mu\rho \, \partial^\mu\rho + \frac{1}{2} \partial_\mu\pi \, \partial^\mu\pi - V(\rho^2 + \pi^2)$$

where the potential is

$$V = \frac{1}{2} \mu^2 \left(\rho^2 + \pi^2\right) + \frac{1}{4} \lambda \left(\rho^2 + \pi^2\right)^2$$

and consider the situation with spontaneous symmetry breaking (SSB), that is, $\mu^2 < 0$. Parameterize the vacuum as

$$\langle\rho\rangle = v\cos\theta \qquad ; \qquad \langle\pi\rangle = v\sin\theta$$

and make the following field redefinition

$$\begin{aligned} \rho =& v\cos\theta + \rho' \\ \pi =& v\sin\theta + \pi' \ . \end{aligned}$$

a) Find the relation between, v, λ, and μ^2 at the SSB minimum.

b) Show that in the field basis, ρ', π', the mass matrix is given at the SSB minimum by,

$$\mathcal{M} = \begin{bmatrix} -2\mu^2\cos^2(\theta) & -\mu^2\sin(2\theta) \\ -\mu^2\sin(2\theta) & -2\mu^2\sin^2(\theta) \end{bmatrix}$$

c) Find and discuss the spectrum of the theory.

2.2 Let us now look at a case where not all the symmetry is broken at the SSB. For this consider a theory with a triplet os scalar fields ϕ^i with $i = 1, 2, 3$. With these fields we construct a theory that is invariant for $O(3)$, that is rotations in the internal symmetry space. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\, \partial_\mu \phi^i \; \partial^\mu \phi^i - \frac{1}{2}\, \mu^2 \phi^i \phi^i - \frac{1}{4}\, \lambda \left(\phi^i \phi^i\right)^2$$

where the sum over repeated indices is implied. With the experience gained before it is easy to see that the SSB minimum ($\mu^2 < 0$) corersponds to

$$\langle \phi^i \rangle \langle \phi^i \rangle = -\frac{\mu^2}{\lambda}$$

This condition does not define the direction of the symmetry breaking. Let us consider that it is the the component ϕ^3 that acquires a vev, that is

$$\langle \phi \rangle = \begin{bmatrix} 0 \\ 0 \\ v \end{bmatrix}$$

The original symmetry group O(3) has $\frac{1}{2} \times 3 \times (3-1) = 3$ generators. The new fact that appears in this example is that that vacuum still has a non trivial symmetry group. This is the subgroup of O(3) that does not mix the component 3 with the others. It is clear that it is O(2) with $\frac{1}{2} \times 2 \times (2-1) = 1$ generator. In agreement with the Goldstone theorem we should have $3 - 1 = 2$ Nambu-Goldstone massless bosons. Let us see how this happens

a) Parametrize the fields as

$$\phi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ v + \sigma \end{bmatrix}$$

with

$$\langle \xi_1 \rangle = \langle \xi_2 \rangle = \langle \sigma \rangle = 0$$

Show that the SSB minimum corresponds to

$$\mu^2 < 0, \quad v^2 = -\frac{\mu^2}{\lambda}$$

b) Expand the Lagrangian in terms of these fields. Show that you get

$$\mathcal{L} = \frac{1}{2}\, \partial_\mu \sigma \; \partial^\mu \sigma + \frac{1}{2}\, \partial_\mu \xi_i \; \partial^\mu \xi_i - \frac{1}{2}\, \mu^2 (v+\sigma)^2 - \frac{1}{4}\, \lambda (v+\sigma)^4$$
$$+ \text{ higher order terms}$$

c) Describe the spectra of the theory, and show that the Goldstone theorem is obeyed.

d) To show that there is nothing special about the third direction, repeat the problem with the following parameterization

$$\phi = \begin{bmatrix} v\sin\theta\cos\varphi + \xi_1 \\ v\sin\theta\sin\varphi + \xi_2 \\ v\cos\theta + \xi_3 \end{bmatrix}$$

where $\langle \xi_i \rangle = 0$. Show that you find the same spectrum.

3 Higgs Mechanism and the Standard Model

3.1 In this problem we want to go through the steps that lead, in the Standard Model, to the expressions of the gauge boson masses after SSB. We start with Lagrangian invariant under SU(2) × U(1) local gauge transformations.

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi^\dagger\phi) - \frac{1}{4}\, W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}\, B_{\mu\nu}B^{\mu\nu}$$

where V is

$$V(\phi^\dagger\phi) = \mu^2\phi^\dagger\phi + \lambda\left(\phi^\dagger\phi\right)^2$$

W^a_μ, $(a = 1, 2, 3)$ and B_μ are the gauge fields corresponding to SU(2) and U(1), respectively. The field tensors are then

$$W^a_{\mu\nu} = \partial_\mu W^a_\mu - \partial_\nu W^a_\mu - g\varepsilon^{abc}W^b_\mu W^c_\nu, \quad B_{\mu\nu} = \partial_\mu B_\mu - \partial_\nu B_\mu$$

The covariant derivative for the Higgs doublet with hypercharge, $Y = 1$, $(Q = T_3 + \frac{1}{2}Y)$, is (we follow the sign conventions of Ref.[1] with all η's positive)

$$D_\mu\phi \equiv \left(\partial_\mu + igW^a_\mu\frac{\tau^a}{2} + ig'B_\mu\frac{1}{2}\right)\phi$$

We want to look at the SSB case that happens when $\mu^2 < 0$ and

$$v^2 = -\frac{\mu^2}{2\lambda}$$

a) Using the Pauli matrices, show that we can write

$$D_\mu\phi = \begin{bmatrix} \partial_\mu + i\frac{g}{2}W^3_\mu + i\frac{g'}{2}B_\mu & i\frac{g}{2}\left(W^1_\mu - i\,W^2_\mu\right) \\ i\frac{g}{2}\left(W^1_\mu + i\,W^2_\mu\right) & \partial_\mu - i\frac{g}{2}W^3_\mu + i\frac{g'}{2}B_\mu \end{bmatrix} \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix}$$

b) With the experience gained in previous examples we can choose a particular gauge, called the *unitary gauge*, where

$$\phi(x) = \begin{bmatrix} 0 \\ v + \frac{\sigma}{\sqrt{2}} \end{bmatrix}, \quad \langle\phi\rangle = \begin{bmatrix} 0 \\ v \end{bmatrix}$$

Show that we get for this case,

$$D_\mu \phi = \begin{bmatrix} i\frac{g}{2}\left(W_\mu^1 - i\,W_\mu^2\right)\left(v + \frac{\sigma}{\sqrt{2}}\right) \\ \frac{\partial_\mu \sigma}{\sqrt{2}} + \left(-i\frac{g}{2}W_\mu^3 + i\frac{g'}{2}B_\mu\right)\left(v + \frac{\sigma}{\sqrt{2}}\right) \end{bmatrix}$$

Warning: Note that here the vev is, $v = \frac{246}{\sqrt{2}}$GeV, and in other problems for the multi-Higgs part is defined as $v = 246$GeV, without the $\sqrt{2}$.

c) Show that substituting the previous expression in the Lagrangian we obtain,

$$\begin{aligned}
\left(D_\mu \phi^\dagger\right)\left(D^\mu \phi\right) =& \frac{1}{2}\,\partial_\mu \sigma\,\partial^\mu \sigma + \left(v^2 + \frac{1}{2}\,\sigma^2 + \sqrt{2}v\sigma\right)\left[\frac{1}{4}\,g^2\left(W_\mu^1 W^{1\mu} + W_\mu^2 W^{2\mu}\right)\right] \\
&+ \left(v^2 + \frac{1}{2}\,\sigma^2 + \sqrt{2}v\sigma\right)\left[\frac{1}{4}\,\left(gW_\mu^3 - g'B_\mu\right)\left(gW^{3\mu} - g'B^\mu\right)\right] \\
=& \frac{1}{2}\,\partial_\mu \sigma\,\partial^\mu \sigma + \frac{1}{4}\,(gv)^2\left(W_\mu^1 W^{1\mu} + W_\mu^2 W^{2\mu}\right) \\
&+ \frac{1}{4}\,v^2\left(gW_\mu^3 - g'B_\mu\right)\left(gW^{3\mu} - g'B^\mu\right) \\
&+ \text{higher order terms} \\
V(\phi^*\phi) =& \text{constant} + \frac{1}{2}\,(-2\mu^2)\sigma^2 + \text{higher order terms} \\
-\frac{1}{4}\,W_{\mu\nu}^a W^{a\mu\nu} =& -\frac{1}{4}\left(\partial_\mu W_\nu^a - \partial_\nu W_\mu^a\right)\left(\partial^\mu W^{a\nu} - \partial^\nu W^{a\mu}\right) + \text{higher order terms} \\
-\frac{1}{4}\,B_{\mu\nu}B^{\mu\nu} =& -\frac{1}{4}\left(\partial_\mu B_\nu - \partial_\nu B_\mu\right)\left(\partial^\nu B^\mu - \partial^\nu B^\mu\right)
\end{aligned}$$

d) Show that there is one scalar field with mass

$$m_\sigma = \sqrt{-2\mu^2}$$

e) Show that in the basis W_μ^3 and B_μ, we have the mass matrix,

$$M^2 = \frac{1}{2}\,v^2\begin{bmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{bmatrix}$$

f) Show that the eigenvalues are 0 and $\frac{1}{2}\,v^2(g^2 + g'^2)$.

g) Call the massless eigenvalue A_μ and the other Z_μ. We can write

$$\begin{cases} A_\mu &= \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu \\ Z_\mu &= \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu \end{cases}$$

The angle θ_W is determined by the requirement that A_μ is the eigenvector with zero mass. Use this to show that

$$\tan\theta_W = \frac{g'}{g}$$

h) Show that the mass of the charged Gauge bosons

$$W_\mu^\pm = \frac{W_\mu^1 \mp iW_\mu^2}{\sqrt{2}}$$

is

$$M_W = \frac{1}{\sqrt{2}} g v$$

and that

$$\frac{M_W}{M_Z} = \cos\theta_W$$

3.2 Obtain the covariant derivative for the Higgs doublet in terms of the physical fields.

a) Show that we can write

$$\begin{aligned} D_\mu\phi &= \left(\partial_\mu + igW_\mu^a\frac{\tau^a}{2} + ig'B_\mu\frac{1}{2}\right)\phi \\ &= \left[\partial_\mu + i\frac{g}{\sqrt{2}}\,W_\mu^+\tau^+ + \frac{g}{\sqrt{2}}\,W_\mu^-\tau^- \right.\\ &\quad \left. + ig\sin\theta_W Q A_\mu + i\frac{g}{\cos\theta_W}\left(\frac{\tau_3}{2} - \sin^2\theta_W Q\right)Z_\mu\right]\phi \end{aligned}$$

where

$$\tau^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \tau^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

b) Use the last expression to identify

$$g\sin\theta_W = e$$

3.3 Show that the covariant derivatives for one family of leptons is

$$D_\mu E_L = \left(\partial_\mu + ig\frac{\tau^a}{2}W_\mu^a - i\frac{g'}{2}B_\mu\right)E_L$$

$$= \left[\partial_\mu + \frac{ig}{\sqrt{2}}\left(W_\mu^+\tau^+ + W_\mu^-\tau^-\right) + ieQA_\mu + i\frac{g}{\cos\theta_W}\left(\frac{\tau^3}{2} - \sin^2\theta_W\,Q\right)Z_\mu\right]E_L$$

$$D_\mu e_R = (\partial_\mu - ig'B_\mu)e_R = (\partial_\mu + ieQA_\mu + ie\tan\theta_W\,Z_\mu)\,e_R$$

3.4 Use the expressions of Problem 3.3 to find the interactions of the first lepton family with the gauge bosons.

$$\begin{aligned}\mathcal{L}_{\text{int}} =& -\frac{g}{2\sqrt{2}}\,\overline{\nu}_e\gamma^\mu(1-\gamma_s)e\,W_\mu^+ - \frac{g}{2\sqrt{2}}\,\overline{e}\gamma^\mu(1-\gamma_5)\nu_e\,W_\mu^-\\ &-\frac{g}{4\cos\theta_W}\Big[\overline{\nu}_e\,\gamma^\mu(1-\gamma_5)\nu_e - \overline{e}\gamma^\mu(1+4Q_e\sin^2\theta_W\ -\gamma_5)e\Big]Z_\mu\\ &-(e\,Q_e)\,\overline{e}\gamma^\mu e\,A_\mu\end{aligned}$$

where $Q_e=-1$ for the electron.

3.5) In this problem we discuss how to give mass to the quarks.

a) Show that an interaction of the form

$$\mathcal{L}_{\text{Yukawa}} = -h_d\,\overline{Q}_L\phi\,d_R + \text{h.c.}$$

gives

$$\mathcal{L}_{\text{Yukawa}} = -h_d\,v\,(\overline{d}_Ld_R+\overline{d}_Rd_L)+\cdots$$

and we get mass for the down quark.

b) Show that we can not use the same mechanism for up quarks, because

$$Y(\overline{Q}_L\phi\,u_R) = -\frac{1}{3}+1+\frac{4}{3} = +2$$

and therefore the term $(\overline{Q}\phi\,u_R)$ is not $SU_L(2)\times U_Y(1)$ invariant.

c) How to solve the problem? In an $SU_L(2)\times U_Y(1)$ transformation we have for the Higgs doublet,

$$\delta\phi = i\varepsilon^a\frac{\tau^a}{2}\phi \qquad\qquad SU_L(2)$$

$$\delta\phi = i\frac{\varepsilon}{2}\phi \qquad\qquad U_Y(1)$$

Consider now the doublet $\tilde{\phi}$ defined by

$$\tilde{\phi} = i\tau_2\phi^* = \begin{pmatrix}\phi^0\\ -\phi^-\end{pmatrix} \qquad ; \qquad \phi^-\equiv(\phi^+)^*$$

Show that for $SU_L(2)$

$$\begin{aligned}\delta\tilde{\phi} =& i\tau_2\ (\delta\phi)^+\\ =& i\tau_2\left(-i\epsilon^a\frac{\tau^a}{2}\,\phi^*\right)\\ =& \epsilon^a\tau_2\,\tau^{a*}\,\frac{1}{2}\,\phi^*\end{aligned}$$

d) Verify the identity

$$\tau_2\, \tau^{a*}\, \tau_2 = \tau^a$$

e) Use the last identity to obtain

$$\delta\tilde{\phi} = i\epsilon^a\, \frac{\tau^a}{2}\, \tilde{\phi}$$

that is, it transforms as ϕ.

f) Show that in an $U_Y(1)$ transformation

$$\delta\tilde{\phi} = i\tau_2\, (\delta\phi)^* = -i\frac{\epsilon}{2}\, \tilde{\phi}$$

This shows that $\tilde{\phi}$ has weak hypercharge -1.

g) Therefore a term

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}} &= -\, h_u\, \overline{Q}_L \tilde{\phi}\, u_R + \text{h.c.} \\ &= -\, h_u v\, (\, \overline{u}_L\, u_R + \overline{u}_R\, u_L) + \cdots\end{aligned}$$

is $SU_L(2) \times U_Y(1)$ invariant, because

$$Y\left(\overline{Q}_L \tilde{\phi}\, u_R\right) = -\frac{1}{3}\, - 1 + \frac{4}{3}\, = 0$$

and can give mass to the quark u.

h) The most general Lagrangian for the mass of the quarks is therefore

$$\mathcal{L}_{\text{Yukawa}} = -\sum_{i,j} h_{dij}\, \overline{Q}_L(i)\phi\, d_R(j) - \sum_{i,j} h_{uij}\, \overline{Q}_L(i)\tilde{\phi}\, u_R(j) + \text{h.c.}$$

There is therefore a mass matrix for the down quarks and another for the up quarks

i) Show that you can diagonalize independently these mass matrices and pass the effect to the interaction terms. Show that nothing happens for the neutral current because it connects only up or down quarks.

j) However the same is not true for the charged currents as it connects up and down quarks. The result is a mixing matrix known as the Cabibbo-Kobayashi-Maskawa matrix

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

that has three angles and one phase as independent parameters for three generations of quarks.

4 Multi-Higgs Models

4.1 This problem shows how to get the mass of the gauge bosons in a 2HDM. Consider a model with two doublets with the same hypercharge ($Y = 1$) that acquire vevs

$$\langle \phi_1 \rangle = \begin{bmatrix} 0 \\ \dfrac{v_1}{\sqrt{2}} \end{bmatrix}, \quad \langle \phi_2 \rangle = \begin{bmatrix} 0 \\ \dfrac{v_2}{\sqrt{2}} \end{bmatrix}$$

a) Show that we can write

$$D_\mu \phi_i|_{\phi_i = \langle \phi_i \rangle} = \begin{bmatrix} i\dfrac{g}{\sqrt{2}} W^+_\mu \dfrac{v_i}{\sqrt{2}} \\ \left(-i\dfrac{g}{2} W^3_\mu + i\dfrac{g'}{2} B_\mu\right) \dfrac{v_i}{\sqrt{2}} \end{bmatrix}$$

b) Use this expression to show that in the basis W^3_μ and B_μ, we have the mass matrix,

$$M^2 = \frac{1}{4}\left(v_1^2 + v_2^2\right) \begin{bmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{bmatrix}$$

and therefore we get the same result as in the SM with the definition

$$v \equiv \sqrt{v_1^2 + v_2^2} \simeq 246\text{GeV}$$

c) Show that the mass of the W is given by

$$M_W = \frac{1}{2} g \sqrt{v_1^2 + v_2^2} \equiv \frac{1}{2} g v$$

d) Genelarize for N Higgs doublets.

4.2 This and the next problem show how to evaluate the ρ parameter in arbitrary Higgs representation. In this problem we consider that in addition the SM one adds one Higgs in the quadruplet representation, with $T = \frac{3}{2}$ and $Y = -1$ with our usual convention that $Q = T_3 + \frac{Y}{2}$. As before the covariant derivative reads

$$D_\mu = \partial_\mu + ig(T_+ W^+_\mu + T_- W^-_\mu) + igW^3_\mu T_3 + ig' Y B_\mu$$

and the fields satisfy the usual relation,

$$\begin{aligned} Z_\mu =& c_W\, W^3_\mu - s_W\, B_\mu\ , \\ A_\mu =& s_W\, W^3_\mu + c_W\, B_\mu\ , \end{aligned}$$

where $g\, s_W = g'\, c_W = e$ and $c_W = \cos\theta_W, s_W = \sin\theta_W$. In the quadruplet representation of SU(2) we have

$$T_3 = \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix}, \quad T_+ = \begin{pmatrix} 0 & \sqrt{3/2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3/2} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_- = T_+^T. \tag{1}$$

a) Show that

$$T_+T_- + T_-T_+ = T^2 - T_3^2.$$

b) Consider now a quadruplet Φ of Higgs fields, with $Y = -1$ and vacuum expectation value

$$\langle 0 \,|\Phi|\, 0\rangle = \begin{pmatrix} 0 \\ V \\ 0 \\ 0 \end{pmatrix}.$$

Obviously the only non-vanishing component has $Q = T_3 + \frac{Y}{2} = 0$.

c) Show that this vev gives a contribution $M_W^2 = (7/2)g^2V^2$ to the $W^\pm$ mass and a contribution $M_Z^2 = g^2V^2/(2c_W^2)$ to the Z mass, thus violating the SM relation $M_W^2 = c_W^2 M_Z^2$.

4.3 Now we want to generalize the previous result. In the SU(2) representation with weak isospin T we have

$$T_3 = \mathrm{diag}\,(T, T-1, T-2, \ldots, -T+1, -T)$$

and

$$(T_+)_{T_3,T_3'} = \delta_{T_3-1,T_3'}\sqrt{\frac{(T-T_3+1)\,(T+T_3)}{2}}, \quad T_- = T_+^T$$

cf a eq. (1). [For instance $(T_+)_{T,T-1} = \sqrt{T}$ and $(T_+)_{T-1,T-2} = \sqrt{2T-1}$.]

a) The gauge bosons mass terms come from the following terms

$$(D_\mu\Phi)^\dagger D^\mu\Phi \supset (0,\ldots,V,\ldots,0)\left[g(T^+W_\mu^+ + T^-W_\mu^-) + gT^3W_\mu^3 + g'\frac{Y}{2}B_\mu\right]$$
$$\left[g(T^+W^{+\mu} + T^-W^{-\mu}) + gT^3W^{3\mu} + g'\frac{Y}{2}B^\mu\right]\begin{bmatrix} 0 \\ \vdots \\ V \\ \vdots \\ 0 \end{bmatrix}$$

Note that T^i are the generators with the correct normalization. We have $T^i = \frac{\tau^i}{2}$ for doublets. Use this to show that you get

$$\mathcal{L}_\mathrm{M} = V^2g^2\left[(T(T+1) - T_3^2\right]W_\mu^+W^{-\mu} + V^2\left(g^2+g'^2\right)\frac{Y^2}{4}Z_\mu Z^\mu$$

b) Show that if an Higgs multiplet with weak isospin T and hypercharge Y has a non-vanishing vev in its conponent with $T_3 = -\frac{Y}{2}$ (that is in its neutral component as we have $Q = T_3 + \frac{Y}{2}$), then the relation $M_W^2 = c_W^2 M_Z^2$ is preserved if and only if $T\,(T+1) = \frac{3}{4}Y^2$.

c) Verify that the previous result is verified for any doublet.

d) Use the previous result to derive the expression for the ρ parameter given in class.

$$\rho = \frac{\sum_i \left[4T_i(T_i+1) - Y_i^2\right] |v_i|^2}{\sum_i 2Y_i^2 |v_i|^2}$$

for complex representations.

e) Verify the result of the previous problem.

f) Show that a septet with $Y = 4$ also has $\rho = 1$.

4.4 This problem evaluates the couplings of the neutral scalars to the gauge bosons in the 2HDM. Consider the model described in Problem 4.1, with the following conventions

$$\phi_1 = \begin{bmatrix} \phi_1^+ \\ \frac{1}{\sqrt{2}}(v_1 + \rho_1 + i\eta_1) \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} \phi_2^+ \\ \frac{1}{\sqrt{2}}(v_2 + \rho_2 + i\eta_2) \end{bmatrix}$$

As we want just to obtain the couplings of the neutral scalars to the gauge bosons, consider $\phi_i^+ = 0$ in the previous expressions

a) Show that in these conditions

$$D_\mu \phi_i \supset \begin{bmatrix} i\frac{g}{\sqrt{2}} W_\mu^+ \frac{1}{\sqrt{2}}(v_i + \rho_i + i\eta_i) \\ i\frac{g}{c_W}(-\frac{1}{2}) Z_\mu \frac{1}{\sqrt{2}}(v_i + \rho_i + i\eta_i) \end{bmatrix}$$

b) Use the previous result to show that we get

$$\mathcal{L} \supset \sum_i \frac{g^2}{2} v_i W_\mu^+ W^{-\mu} \rho_i + \sum_i \frac{g^2}{c_W^2} \frac{1}{4} v_i Z_\mu Z^\mu \rho_i$$

c) In class we saw that, $v_1 = vc_\beta, v_2 = vs_\beta$, where β is the angle that changes the original basis into the Higgs basis ($\tan\beta = v_2/v_1$). Also

$$\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} c_\alpha & -s_\alpha \\ s_\alpha & c_\alpha \end{bmatrix} \begin{bmatrix} h \\ H \end{bmatrix}$$

Show that

$$\sum_i v_i \rho_i = v\left[\cos(\beta - \alpha)h + \sin(\beta - \alpha)H\right]$$

and therefore

$$\mathcal{L} \supset \left[g M_W W_\mu^+ W^{-\mu} + \frac{g M_Z}{2c_W} Z_\mu Z^\mu \right] (C_h h + C_H H)$$

where

$$C_h = \cos(\beta - \alpha), \quad C_H = \sin(\beta - \alpha)$$

d) Show that there is no trilinear coupling of the pseudoscalar A with the gauge bosons.

4.5 Type I couplings in the 2HDM.
Consider that we are in the basis of the mass eigenstates for the quark and leptons, that is we have passed the mismatch of the diagonalizations to the gauge interactions (CKM matrix). Then in Type I couplings all the fermions couple to the same Higgs field that we take conventionaly as ϕ_2. Then the corresponding Yukawa Lagrangian reads

$$\mathcal{L}_{\rm Yuk} = -\frac{\sqrt{2}M_d}{v_2}\overline{Q}_L\phi_2 d_R - \frac{\sqrt{2}M_u}{v_2}\overline{Q}_L\tilde{\phi}_2 u_R - \frac{\sqrt{2}M_\ell}{v_2}\overline{L}_L\phi_2 e_R + \text{h.c.}$$

where

$$M_d = \text{diag}(m_d, m_s, m_b), \quad M_u = \text{diag}(m_u, m_c, m_t), \quad M_\ell = \text{diag}(m_e, m_\mu, m_\tau)$$

and (see problem 3.5)

$$\tilde{\phi}_2 = i\tau_2\,\phi_2^* \,.$$

a) Show that for any fermion (ψ_f)we can write the Yukawa interaction as

$$\mathcal{L}_Y = -\frac{m_f}{v}\overline{\psi}_f\left(a_i^f + i\,b_i^f\gamma_5\right)\psi_f h_i$$

where $h_i = h, H, A$ for the 2HDM, and for Type I

$$\begin{aligned}
a_h^u = a_h^d = a_h^\ell &= \frac{s_\alpha}{s_\beta}, & b_h^u = b_h^d = b_h^\ell &= 0\\
a_H^u = a_H^d = a_H^\ell &= \frac{c_\alpha}{s_\beta}, & b_H^u = b_H^d = b_H^\ell &= 0\\
a_A^u = a_A^d = a_A^\ell &= 0, & b_A^u = -\frac{1}{t_\beta},\ b_A^d = b_A^\ell &= \frac{1}{t_\beta}
\end{aligned}$$

b) Why has b_A^u the opposite sign of $b_A^d = b_A^\ell$?

4.6 Type II couplings in the 2HDM
In this case the up quarks couple to ϕ_2 and down quarks and leptons to ϕ_1. The Yukawa Lagrangian is now given by,

$$\mathcal{L}_{\rm Yuk} = -\frac{\sqrt{2}M_d}{v_1}\overline{Q}_L\phi_1 d_R - \frac{\sqrt{2}M_u}{v_2}\overline{Q}_L\tilde{\phi}_2 u_R - \frac{\sqrt{2}M_\ell}{v_1}\overline{L}_L\phi_1 e_R + \text{h.c.}$$

a) Show that for this case we have

$$\begin{aligned}
a_h^u = \frac{s_\alpha}{s_\beta},\ a_h^d = a_h^\ell &= \frac{c_\alpha}{c_\beta}, & b_h^u = b_h^d = b_h^\ell &= 0\\
a_H^u = \frac{c_\alpha}{s_\beta},\ a_H^d = a_H^\ell &= -\frac{s_\alpha}{c_\beta}, & b_H^u = b_H^d = b_H^\ell &= 0\\
a_A^u = a_A^d = a_A^\ell &= 0, & b_A^u = -\frac{1}{t_\beta},\ b_A^d = b_A^\ell &= -t_\beta
\end{aligned}$$

b) Is the origin of the minus sign in b_A^u the same as the minus sign in $b_A^d = b_A^\ell$?

5 Higgs Phenomenology at the LHC

5.1 Suppose that 156 Higgs are collected with a luminosity of 130 fb^{-1} in the process $gg \to h$ followed by $h \to b\bar{b}$. In a new model (NM) with a simple extension of the Higgs potential the cross section is scaled as $\sigma^{NM}(gg \to h)/\sigma^{SM}(gg \to h) = \sin^2\alpha$ and the branching ratio is approximately scaled as $BR^{NM}(h \to b\bar{b})/BR^{SM}(h \to b\bar{b}) \approx \sin^2\alpha$. The expected number of Higgs events for the SM Higgs and for the same luminosity is 10 % more in the SM than in the NM. Find the value of α in the domain $[-\pi/2, \pi/2]$.

5.2 The gluon fusion cross section for the production of a 400 GeV Higgs in model A is 196 fb. The total width for this 400 GeV Higgs is 26 GeV while the partial width for the decay to tau leptons is 2 GeV. How many $\tau^+\tau^-$ pairs will be produced for a luminosity of 300 fb^{-1}?

5.3 The gluon fusion cross section for the production of a 400 GeV Higgs in model A is 196 fb. The total width for this 400 GeV Higgs is 26 GeV while the partial decay for two SM Higgs is 10 GeV. The two SM Higgs decay to pairs of b-quarks. The complete process is $gg \to H \to h_{125}h_{125} \to \bar{b}b\bar{b}b$. How many b-quarks and how many anti-b quarks will be detected for a luminosity of 3000 fb^{-1}? (Consider that the $BR(h_{125} \to \bar{b}b) = 70\%$).

References

[1] J. C. Romao and J. P. Silva, *A resource for signs and Feynman diagrams of the Standard Model*, *Int. J. Mod. Phys.* **A27** (2012) 1230025, [1209.6213].