Exercises

* What are n, u, P for relativistic degenerate fermion gas?

$$dN = gV \frac{d^3k}{(2\pi)^3} \Rightarrow n = \frac{k_F^3}{3\pi^2}$$

$$u = \int \frac{2}{8\pi^3} \underbrace{\overset{=1}{c}k}_{\text{relativistic dispersion}} 4\pi k^2 \, dk = \frac{k_F^4}{4\pi^2}$$

$$P = \frac{2}{3} \int k \cdot \overset{=1}{c} \frac{k^2}{2\pi^2} dk = \frac{1}{3\pi^2} \frac{k_F^4}{4} = \frac{1}{3} u$$

$$\boxed{P = \frac{1}{3} u}$$

* Lane-Emden solutions

n=0

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta_0}{d\xi}\right) = -\theta^0 = -1$$

$$\int \frac{d}{d\xi}\left(\xi^2\frac{d\theta_0}{d\xi}\right)d\xi = \int -\xi^2 d\xi \Rightarrow$$

$$\xi^2\frac{d\theta_0}{d\xi} = c_1 - \frac{\xi^3}{3} \Rightarrow$$

$$\frac{d\theta_0}{d\xi} = \frac{c_1}{\xi^2} - \frac{\xi}{3}$$

c_1 must be zero because otherwise

$\frac{d\theta_0}{d\xi} \to \infty$ as $\xi \to 0$

$$\theta_0 = c_2 - \frac{\xi^2}{6} \qquad \underline{\theta_0(\xi=0) = 1}$$

$$\theta_0(\xi) = 1 - \frac{\xi^2}{6} \qquad \xi_1 = \sqrt{6}$$

$$R = \alpha \xi_1$$

$n = 1$ $\quad \frac{d}{d\xi}\left(\xi^2 \frac{d\theta_1}{d\xi}\right) = -\xi^2 \theta_1$

$$\xi^2 \theta_1'' + 2\xi \theta_1' + \xi^2 \theta_1 = 0$$

$$\theta_1'' + \theta_1 + \frac{2}{\xi}\theta_1' = 0$$

Try $\theta_1 = A \sin\xi \cdot \xi^s$

$$A(1+s)\xi^{-2+s}(2\xi\cos\xi + s\sin\xi) = 0$$

For $s = -1$ we get a solution

$$\theta_1 = \frac{\sin\xi}{\xi} \qquad \lim_{\xi \to 0} \frac{\sin\xi}{\xi} = 1 \quad \xi_1 = \pi$$

* Relate M and R without explicit or implicit dependence on ρ_c

$$R = \left[\frac{(n+1) K \rho_c^{\frac{1-n}{n}}}{4\pi G} \right]^{1/2} \xi_1 = \alpha \xi_1$$

$$M = -4\pi \alpha^3 \rho_c \xi_1^2 \left. \frac{d\theta}{d\xi} \right|_{\xi_1}$$

$$\alpha = \left[\frac{(n+1) K \rho_c^{\frac{1-n}{n}}}{4\pi G} \right]^{1/2}$$

$$R \sim \alpha \sim \rho_c^{\frac{1-n}{2n}}$$

$$M \sim \alpha^3 \rho_c \sim \rho_c^{\frac{3-3n}{2n}+1} \sim \rho_c^{\frac{3-n}{2n}}$$

$$\boxed{K = N_n \, G \, M^{\frac{n-1}{n}} \, R^{\frac{3-n}{n}}}$$

This combination is independent of ρ_c

$$N_n = \frac{(4\pi)^{1/n}}{n+1} \left[-\xi_1^{\frac{n+1}{n-1}} \left. \frac{d\theta_n}{d\xi} \right|_{\xi_1} \right]^{\frac{1-n}{n}}$$

$P = K\rho^{5/3}$ non-relativistic $\gamma = 1 + \frac{1}{n} \Rightarrow n = \frac{3}{2}$

$P = K\rho^{4/3}$ relativistic $n = 3$

$$K = N_{3/2} \, G \, M^{\frac{1/2}{3/2}} R = N_{3/2} \, G M^{1/3} R$$

$M^{1/3} R = \text{const.} \Rightarrow \boxed{M \sim \frac{1}{R^3}}$

$$K = N_3 \, G \, M^{\frac{2}{3}} R^{0} = N_3 \, G \, M^{2/3}$$

$$M = \text{const.} \qquad E = \sqrt{c^2 k^2 + m^2 c^4}$$

Chandrasekhar mass limit

* Gravitational Collapse of fermions

Non-interacting

$$E = KE + PE = c\, k_F - \frac{GMm}{r} < 0$$

$$n = \frac{k_F^3}{3\pi^2} \Rightarrow k_F = (3\pi^2 n)^{1/3} = \left(\frac{3\pi^{\cancel{2}} N}{\frac{4}{3}\cancel{\pi} r^3}\right)^{1/3} =$$

$$= \left(\frac{9\pi}{4}\right)^{1/3} \frac{N^{1/3}}{r}$$

$$E = \left(\frac{9\pi}{4}\right)^{1/3} \frac{N^{1/3}}{\cancel{r}} - G\frac{Nm^2}{\cancel{r}} < 0$$

$$G N m^2 > \left(\frac{9\pi}{4}\right)^{1/3} N^{1/3} \Rightarrow$$

$$N^{2/3} > \left(\frac{9\pi}{4}\right)^{1/3} \frac{1}{G m^2} \qquad G = \frac{1}{m_{pl}^2}$$

$$N^{2/3} > \left(\frac{9\pi}{4}\right)^{1/3} \frac{m_{pl}^2}{m^2} \Rightarrow$$

$$N > \left(\frac{9\pi}{4}\right)^{\frac{1}{2}} \frac{m_{pl}^3}{m^3} = N_{crit}$$

For $m = 1\,\text{GeV}$ $\quad M_{crit} = N_{crit}\, m = \mathcal{O}(M_\odot)$

Maximum NS mass $\sim 2.4\, M_\odot$

TOV $\quad M = 0.7\, M_\odot$

* Gravitational Collapse of bosons

Uncertainty principle $\Delta x \, \Delta p \geq \hbar$

$$\Delta p \geq \frac{\hbar}{\Delta x} \qquad p_{crit} = \frac{1}{r} \qquad \hbar = c = k_B = 1$$

$$E = KE + PE = \overset{=1}{c}\, p_{crit} - \frac{GMm}{r} =$$

$$= \frac{1}{\cancel{r}} - \frac{GNm^2}{\cancel{r}} < 0$$

$$1 < Nm^2 G \Rightarrow N \geq \frac{m_{pl}^2}{m^2}$$

$N_{crit} = \frac{m_{pl}^2}{m^2}$ bosons $\quad \frac{10^{38}}{m^2} \sim \frac{10^{-19} M_\odot}{m^2}$

$N_{crit} \sim \frac{m_{pl}^3}{m^3}$ fermions $\quad \frac{10^{57}}{m^3} \sim \frac{M_\odot}{m^3}$

* Astrophysical Black holes

$M > 3M_\odot$

Mass of supermassive star $9M_\odot \lesssim M \lesssim 20M_\odot$

NS formation , $M \gtrsim 20M_\odot$ BH formation

* Primordial Black holes

Chemical Equilibrium & Charge Neutrality

$$n \to p + e^- + \bar{\nu}_e$$

$\mu_n = \mu_p + \mu_e$ ① (neutrinos escape)

$n_p = n_e$ (charge neutrality)

$$n_p = \frac{k_{Fp}^3}{3\pi^2}, \quad n_e = \frac{k_{Fe}^3}{3\pi^2} \Rightarrow k_{Fp} = k_{Fe}$$

At zero temperature

$$\mu_p = E_p = m_p + \frac{k_{Fp}^2}{2m_p}$$

proton non-relativistic

$$\mu_e = k_{Fe} \overset{=1}{c} = k_{Fp}$$

electron relativistic

we should check this a posteriori

$$\underbrace{m_n + \frac{k_{Fn}^2}{2m_n}}_{\mu_n} = \underbrace{m_p + \frac{k_{Fp}^2}{2m_p}}_{\mu_p} + \underbrace{k_{Fp}}_{\mu_e} \quad (1)$$

$$\rho_{tot} = n_p m_p + n_n m_n \quad (2) \quad (\text{ignore electrons})$$

$$\frac{k_{Fn}^2}{2m_n} = k_{Fp}\left(1 + \frac{k_{Fp}}{2m_p}\right) - Q$$

$$Q = m_n - m_p \simeq 1.3 \text{ MeV}$$

we assume $\frac{k_{Fp}^2}{m_p}, Q \ll k_{Fp}$

(must be checked later)

$$\frac{k_{Fn}^2}{2m_n} = k_{Fp} \quad (3)$$

(2) $\rho_{tot} \simeq m_n n_n + m_p n_p \simeq m_n n_n$

(must be checked!)

$$n_n = \frac{k_{Fn}^3}{3\pi^2} \qquad n_p = \frac{k_{Fp}^3}{3\pi^2}$$

$$k_{Fn} = \left(3\pi^2 n_n\right)^{1/3}$$

(2) $\Rightarrow$ $\rho_{tot} \simeq n_n m_n \Rightarrow n_n \simeq \frac{\rho_{tot}}{m_n} =$

$$= \frac{\rho_{tot}}{\rho_{nuc}} \frac{\rho_{nuc}}{m_n}$$

$$\rho_{nuc} \simeq 2.8 \times 10^{14} \, g/cm^3$$

$$n_n \simeq 1.7 \times 10^{38} \, cm^{-3} \left(\frac{\rho_{tot}}{\rho_{nuc}}\right)$$

$$n_p = n_e = \frac{k_{Fp}^3}{3\pi^2} \overset{3}{\simeq} \left(\frac{k_{Fn}^2}{2m_n}\right)^3 \frac{1}{3\pi^2} =$$

$$= \frac{1}{(2m_n)^3} \underbrace{\left(\frac{k_{Fn}^3}{3\pi^2}\right)^2}_{n_n} \frac{(3\pi^2)^{\cancel{2}}}{\cancel{3\pi^2}} = \frac{3\pi^2}{8m_n^3}\, n_n^2$$

$$n_p = n_e \simeq 9.5 \times 10^{35}\ \text{cm}^{-3} \left(\frac{\rho_{tot}}{\rho_{nuc}}\right)^2$$

Indeed for $\rho_{tot} \simeq \rho_{nuc}$

$$n_p = n_e \ll n_n$$

$$k_{Fn} = (3\pi^2 n_n)^{1/3} \simeq 340 \left(\frac{\rho_{tot}}{\rho_{nuc}}\right)^{1/3} \frac{\text{MeV}}{c}$$

$$k_{Fp} = k_{Fe} = (3\pi^2 n_p)^{1/3} = 60 \left(\frac{\rho_{tot}}{\rho_{nuc}}\right)^{2/3} \frac{\text{MeV}}{c}$$

$m_p \sim 0.9$ GeV Indeed protons are non-relativistic

$m_e \sim 0.5$ MeV $\mu_{Fe} \simeq 60$ MeV electrons are relativistic

$$n \simeq \frac{1}{e^{\frac{E-\mu}{k_B T}} + 1}$$

$\frac{\mu}{T} >> 1$ cold object

$\frac{\mu}{T} << 1$ hot object

10^{13} K $\sim$ 1 GeV

10^{7} K $\sim$ 1 keV $= T$

$\mu \sim$ GeV