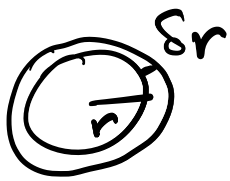
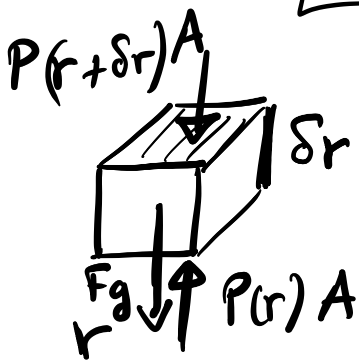
Basic Equations of Stars



$$dM = 4\pi r^2 \rho(r)\, dr$$

$$\boxed{\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)} \quad (1)$$



$$F_{net} = 0 \Rightarrow$$

$$P(r+\delta r)A - P(r)A + F_g = 0$$

$$F_g = \frac{G M(r)\, \delta m}{r^2} = \frac{G M(r) \rho A\, \delta r}{r^2}$$

$$\left[P(r+\delta r) - P(r)\right] \cancel{A} = - \frac{G M \rho \cancel{A}\, \delta r}{r^2} \Rightarrow$$

$$\lim_{\delta r \to 0} \frac{P(r+\delta r) - P(r)}{\delta r} = - \frac{G M \rho}{r^2}$$

$$\boxed{\frac{dP}{dr} = -G\frac{M\rho}{r^2}} \quad (2)$$

Tolman–Oppenheimer–Volkoff Eq.

spherically symmetric metric and static

$$ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2 d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}$$

$$T_{\mu\nu} = P g_{\mu\nu} + (P+\rho) U_\mu U_\nu$$

$$U_\mu = \frac{dx^\mu}{d\tau}$$

$$\frac{dP}{dr} = -\underbrace{\frac{GM\rho}{r^2}}_{\text{Newtonian}} \overbrace{\left[1+\frac{P}{\rho}\right]\left[1+\frac{4\pi r^3 P}{M}\right]}^{GR}$$

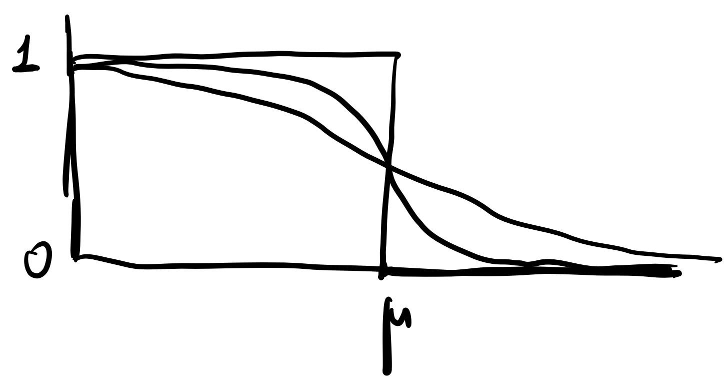
We need an Equation of State

<u>Degenerate matter</u> $\hbar = 1$

$k_B = 1$

$$dN = g V \frac{d^3k}{(2\pi)^3} \times \frac{1}{e^{\frac{E-\mu}{T}}+1}$$

$g \leftarrow 2$



NS $\frac{\mu}{T} \gg 1$ are very cold objects

$$dn = \frac{dN}{V} = \frac{2 \cdot 4\pi k^2 dk}{8\pi^3} = \frac{1}{\pi^2} k^2 dk$$

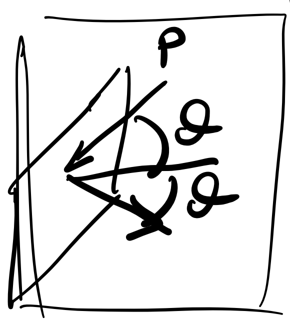
$$n = \frac{k_F^3}{3\pi^2}$$

For non-relativistic $\rho = n \cdot m = \frac{m\, k_F^3}{3\pi^2}$

$$\frac{dE}{V} = du = \frac{1}{\pi} k^2 \frac{k^2}{2m} dk \Rightarrow u = \frac{k_F^5}{10\pi^2 m}$$

$$= \frac{k_F^2}{2m} \cdot \frac{k_F^3}{5\pi^2} = \frac{3}{5} E_F\, n$$

what about the pressure?



L

$$\delta p = 2 p \cos\vartheta$$

$$2L = v\cos\vartheta\, \delta t \Rightarrow$$

$$\delta t = \frac{2L}{v\cos\vartheta}$$

$$\frac{\delta p}{\delta t} = \frac{2 p \cos\vartheta\, v \cos\vartheta}{2L}$$

$$dP = \frac{dF}{A} = \frac{p\, v \cos^2\vartheta}{\cancel{V}}\, n(p)\, \cancel{V}\, d\cos\vartheta$$

$$dn = \frac{1}{4\pi^3} p^2 dp\, 2\pi\, d\cos\vartheta = \underbrace{\frac{p^2 dp}{2\pi^2}}_{n(p)} d\cos\vartheta$$

$$P = \int_{-1}^{1} p\, v \cos^2\vartheta\, n\, d\cos\vartheta\, dp =$$

$$= \frac{2}{3} \int p\, v\, n(p)\, dp$$

$$P = \frac{2}{3} \int_0^{p_F} p \frac{p}{m} \frac{p^2}{2\pi^2} dp = \frac{2\, p_F^5}{6\pi^2 m \cdot 5} = \frac{2}{3} u$$

non-relat.

$$P = \frac{p_F^5}{15\pi^2 m} \qquad \rho = \frac{p_F^3\, m}{3\pi^2} \qquad p_F \sim \rho^{1/3}$$

$$P = K\rho^{\gamma} \quad \text{EoS}$$

$\gamma = \frac{5}{3}$ non-rel.

$\gamma = \frac{4}{3}$ rel.

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \quad \text{hydro}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho \quad \text{mass continuity}$$

derivative with respect to r

$$-GM = \frac{r^2}{\rho}\frac{dP}{dr} \Rightarrow$$

$$-G\frac{dM}{dr} = \frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) \Rightarrow$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G r^2 \rho \Rightarrow$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}K\gamma\rho^{\gamma-1}\frac{d\rho}{dr}\right) = -4\pi G\rho$$

$$\rho \equiv \rho_c\theta^n \qquad \gamma \equiv 1+\frac{1}{n} \Rightarrow n = \frac{1}{\gamma-1}$$

$$\frac{K\left(1+\frac{1}{n}\right)}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho_c^{\frac{1-n}{n}}\cancel{\theta^{1-n}}\,n\cancel{\rho_c}\cancel{\theta^{n-1}}\right.$$

$$\left.\times\frac{d\theta}{dr}\right) = -\cancel{\rho_c}\theta^n \Rightarrow$$

$$\frac{(n+1)K}{4\pi G}\rho_c^{\frac{1-n}{n}}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n$$

$$r \equiv \alpha\xi \qquad \alpha^2 = \frac{(n+1)K\rho_c^{\frac{1-n}{n}}}{4\pi G}$$

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n}$$

Lane - Emden Equation

Initial condition $\theta_n = 1$ for $\xi = 0$

$\theta_n(\xi) = 0$ $\quad \xi_1$ (first zero)

$r = a\xi$ $\quad \overset{\text{radius of star}}{R} = a\xi_1$

Analytic solutions exist for
$n=0$, $n=1$ and $n=5$

$$M = \int 4\pi r^2 \rho \, dr = 4\pi \rho_c a^3 \int_0^{\xi} \xi^2 \theta^n \, d\xi$$

$$= -4\pi a^3 \rho_c \int_0^{\xi} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi$$

$$= -4\pi a^3 \rho_c \xi^2 \frac{d\theta}{d\xi}$$

$$M_{star} = -4\pi a^3 \rho_c \xi_1^2 \frac{d\theta}{d\xi}\Big|_{\xi_1}$$

Exercises

- Find n, u, and P for relativistic degenerate fermions (i.e., $E = cp$).
- Solve the Lane-Emden equation for $n = 0$ and $n = 1$.
- Use $R_{star} = a\xi_1$ and $M_{star} = -4\pi a^3 \rho_c \xi_1^2 \frac{d\theta}{d\xi}\Big|_{\xi_1}$ to eliminate any dependence on ρ_c. How M and R are related in the case of non-relativistic and relativistic degenerate fermions?