

Renormalization group improved pressure for QCD

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Overview

1 Introduction

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- QCD perturbative pressure
- Renormalization group
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QCD characteristics

QCD Lagrangian

$$\mathcal{L}_{\text{QCD}} = \bar{\psi}_i (i(\gamma^\mu D_\mu) - m_i) \psi_i - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu}$$

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$$

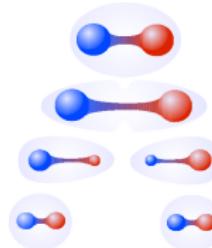
$$D = \partial_\mu - ig T^a A_\mu^a$$

$\{T^a; a = 1, \dots, 8\} \rightarrow SU(3)$ generators

$A_\mu^a(x) \rightarrow$ Gluon fields

$g \rightarrow$ QCD coupling constant.

- Confinement.
- Asymptotic freedom.
- Non-perturbative behavior at low energies.



$$\alpha_s(Q^2) \equiv \frac{g^2(Q^2)}{4\pi} = \frac{4\pi}{(11 - \frac{2}{3}N_f) \ln(Q^2/\Lambda_{QCD}^2)}$$

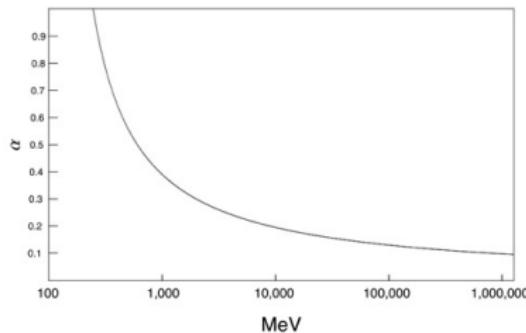
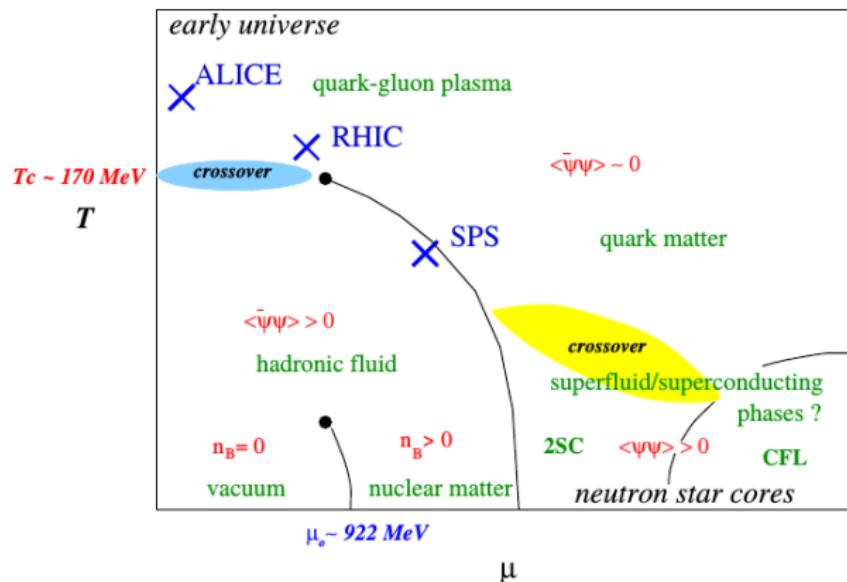


Diagrama de fase da QCD



Methods

Perturbative QCD (pQCD)

Based on the asymptotic freedom, only valid in the high energy regime.

$1/N_c$ expansion

An expansion in $1/N_c$ is made, where N_c is treated as infinity, at the end corrections are made to reconcile the fact that $N_c = 3$.

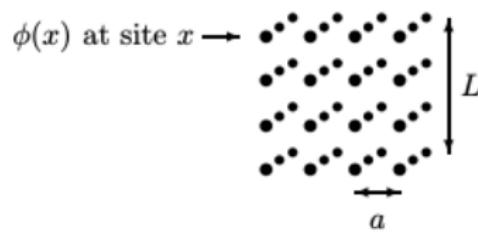
Lattice QCD (LQCD)

Discrete point network used to calculate path integrals. Calculations are performed on supercomputers.

Effective theories

Used in the low energy regime, where there are parameters that are not in the original theory..

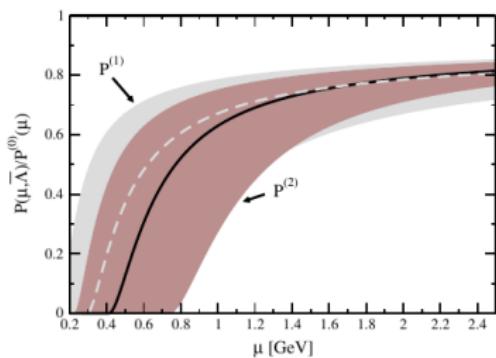
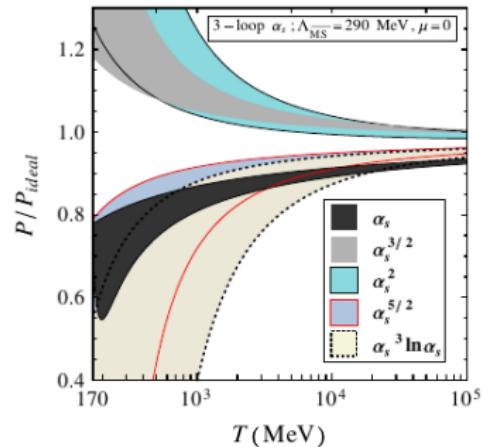
Lattice QCD



Sign problem

$\det(\gamma_\mu D^\mu + m - \gamma_0 \mu) \rightarrow$ Complex number.

perturbative QCD



Non-perturbative QCD approximations

Large N (LN) approximation (mean field, Hartree).

Landau theorem is violated.

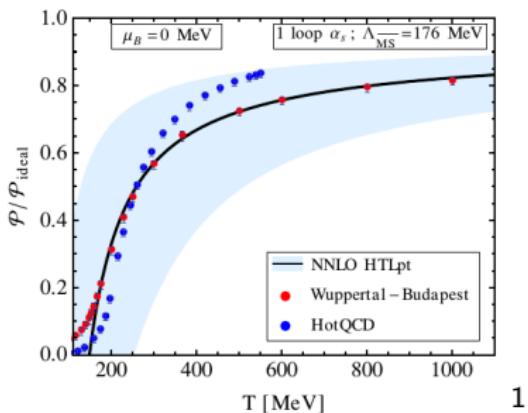
fails to localized the critic point in the liquid gas transition for fermionic theories in $2 + 1$ dimensions, etc.

resume techniques (OPT, SPT e HTLpt)

Poor convergence.

High regularization scale dependence.

Hard Thermal Loops Perturbation Theory (HTLpt)



- Results close to QCD when $M = 2\pi T$.
- High temperature approximation $m/T \ll 1$.
- Can not see transition.

¹Haque et al (2014)

RGOPT

OPT modified in order to satisfy renormalization group properties.

Already used to calculate:

The QCD scale, $\Lambda_{\overline{\text{MS}}}$. [Kneur and Neveu (2012)]

The QCD coupling constant, α_s . [Kneur and Neveu (2013)]

The quark condensate from the spectral function. [Kneur and Neveu (2015)]

Applied in scalar models at finite temperature. [Kneur and Marcus Benghi (2015), Ferrari et al (2017)]

QCD perturbative pressure

$$P^{PT} = \text{Diagram A} + \text{Diagram B} + \mathcal{O}(g^2)$$

$$\text{Diagram A} = -i \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \ln (\not{p} - m_f) ,$$

$$\begin{aligned} \text{Diagram B} &= -g (4\pi)^2 \frac{(N_c^2 - 1)}{2} \int \frac{dp^4}{(2\pi)^4} \frac{dq^4}{(2\pi)^4} \frac{dk^4}{(2\pi)^4} (2\pi)^4 \delta^4(k - p + q) \\ &\times \text{Tr} \left[\frac{\gamma^\mu (\not{p} + m_f) \gamma_\mu (\not{q} + m_f)}{k^2 (p^2 - m_f^2) (q^2 - m_f^2)} \right], \end{aligned}$$

where $g = \frac{g_s^2}{(4\pi)^2}$

$$\frac{P_{1,f}^{PT}}{N_c} = -\frac{m_f^4}{8\pi^2} \left(\frac{3}{4} - L_m \right) + 2T J_1(T, \mu) - 3g \frac{m_f^4}{2(2\pi)^4} C_F \left(L_m^2 - \frac{4}{3}L_m + \frac{3}{4} \right) \\ - gC_F \left\{ \left[\frac{m_f^2}{4\pi^2} (2 - 3L_m) + \frac{T^2}{6} \right] J_2(T, \mu) + \frac{1}{2} J_2^2(T, \mu) + \frac{m_f^2}{2} J_3(T, \mu) \right\},$$

where $L_m = \ln(m_f/M)$, $g = 4\pi\alpha_s$, $C_F = (N_c^2 - 1)/(2N_c)$, $N_c = 3$, and $N_f = 3$.

$$J_1(T, \mu) = \int \frac{d^3 p}{(2\pi)^3} \left\{ \ln \left[1 + e^{-(E_p + \mu)/T} \right] + \ln \left[1 + e^{-(E_p - \mu)/T} \right] \right\},$$

and

$$J_2(T, \mu) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{E_p} \left[f^+(E_p) + f^-(E_p) \right],$$

$$J_3(T, \mu) = \frac{1}{(2\pi)^4} \int_0^\infty \int_0^\infty \frac{dp \, pdq \, q}{E_p E_q} \left\{ \Sigma_+ \ln \left[\frac{E_p E_q - m_f^2 - pq}{E_p E_q - m_f^2 + pq} \right] + \Sigma_- \ln \left[\frac{E_p E_q + m_f^2 + pq}{E_p E_q + m_f^2 - pq} \right] \right\},$$

where

$$\Sigma_\pm = f^+(E_p) f^\pm(E_q) + f^-(E_p) f^\mp(E_q).$$

[Kapusta (1978)]

Renormalization group

Physical quantities must be independent of the arbitrary energy scale M

$$M \frac{d}{dM} G^{(n)} = 0.$$

Renormalization group Equation

$$\left[M \frac{\partial}{\partial M} + \beta \frac{\partial}{\partial g} - \gamma_m m \frac{\partial}{\partial m} - \frac{n}{2} \zeta \right] G^{(n)} (m, g, M) = 0.$$

where

$$\beta = M \frac{dg}{dM},$$

and

$$\gamma_m = -M \frac{d}{dM} \ln m.$$

RG modified Perturbative pressure

The perturbative pressure is not RG invariant-

$$\left[M \frac{\partial}{\partial M} + \beta \frac{\partial}{\partial g} - \gamma_m m \frac{\partial}{\partial m} \right] P_{0,f}^{\text{PT}} = M \frac{\partial}{\partial M} P_{0,f}^{\text{PT}} = -\frac{m^4}{8\pi^2}.$$

To recover RG invariance one must add the terms

$$\frac{\mathcal{E}^{\text{RG}}}{N_c} = m^2 \sum_{k \geq 0} s_k g^{k-1}.$$

$$P^{\text{RGI}} = P^{\text{PT}} + \mathcal{E}^{\text{RG}},$$

RG modified Perturbative pressure

β and γ_m functions

$$\beta = -2b_0g^2 - 2b_1g^3 + \mathcal{O}(g^4)$$

$$\gamma_m = \gamma_0 g + \gamma_1 g^2 + \mathcal{O}(g^3),$$

$$b_0 = \frac{1}{(4\pi)^2} \left(11 - \frac{2}{3} N_f \right),$$

$$b_1 = \frac{1}{(4\pi)^4} \left(102 - \frac{38}{3} N_f \right),$$

$$\gamma_0 = \frac{1}{2\pi^2},$$

$$\gamma_1 = \frac{1}{8(2\pi)^4} \left(\frac{202}{3} - \frac{20}{9} N_f \right).$$

Chetyrkin (2005), Czakon (2005)

RG modified Perturbative pressure

$$\frac{P^{RGI}}{N_f N_c} = \frac{P^{PT}}{N_f N_c} + m^4 \left(\frac{s_0}{g} + s_1 \right).$$

$$s_0 = \frac{(b_0 - 2\gamma_0)^{-1}}{(4\pi)^2}$$

$$s_1 = \frac{1}{4} \left[\frac{b_1 - 2\gamma_1}{4(b_0 - 2\gamma_0)} - \frac{1}{12\pi^2} \right].$$

Renormalization group optimized perturbation theory (RGOPT)

$$\mathcal{L}_{QCD}^{RGOPT} = \mathcal{L}_{QCD}|_{g \rightarrow \delta g} - m(1 - \delta)^a \bar{\psi}_f \psi_f,$$

$$((m_u = m_d = m_s = 0))$$

(Equivalent to replace $m_f \rightarrow m(1 - \delta)^a$ and $g \rightarrow \delta g$ on the perturbative pressure, P^{RGI}).

One must solve simultaneously

$$\left[M \frac{\partial}{\partial M} + \beta \frac{\partial}{\partial g} \right] P = 0 \rightarrow \text{Reduced RG equation}$$

$$\frac{\partial P}{\partial m} \Big|_{\bar{m}} = 0 \rightarrow \text{PMS}$$

Order δ^0

$$\frac{P_{0,f}^{RGOPT}}{N_c} = -\frac{2m^4}{(4\pi)^2} \left(\frac{3}{4} - L_m\right) + 2TJ_1(T, \mu) + \frac{m^4}{g} s_0 (1 - 4a).$$

First one must find a at order δ^0 no vacuo ($T = mu = 0$)

$$f_{\text{RG}} = \left[M \frac{\partial}{\partial M} + \beta \frac{\partial}{\partial g} \right] P_{0,f}^{RGOPT} = 0,$$

↓

$$a = \frac{\gamma_0}{2b_0} \rightarrow \text{same than other theories}$$

$$f_{\text{PMS}} = \frac{\partial P_{0,f}^{RGOPT}}{\partial m} \equiv 0.$$

↓

$$\overline{m}(\mu = 0) = \Lambda_{\overline{\text{MS}}} \sqrt{e}$$

$$g_{1L}(M) = \frac{g(M_0)}{1 + 2g(M_0)b_0 \ln\left(\frac{M}{M_0}\right)}.$$

RGOP order δ

$$\begin{aligned}\frac{P_{1,f}^{RGOPT}}{N_f N_c} &= -\frac{m^4}{8\pi^2} \left(\frac{3}{4} - L_m \right) + 2T J_1(T, \mu) + \frac{m^4}{(2\pi)^2} \left(\frac{\gamma_0}{b_0} \right) \left(\frac{1}{2} - L_m \right) + m^2 \left(\frac{\gamma_0}{b_0} \right) J_2(T, \mu) \\ &- 3g \frac{m^4}{2(2\pi)^4} C_F \left(L_m^2 - \frac{4}{3}L_m + \frac{3}{4} \right) - gC_F \left[\frac{2m^2}{(4\pi)^2} (4 - 6L_m) - \frac{T^2}{6} \right] J_2(T, \mu) + \frac{g}{2} C_F J_3(T, \mu) \\ &+ \frac{m^4}{(4\pi)^2 b_0} \left\{ \frac{1}{g} \left(1 - \frac{\gamma_0}{b_0} \right) + \left[(b_1 - 2\gamma_1) \pi^2 - \frac{(b_0 - 2\gamma_0)}{3} \right] \right\}.\end{aligned}$$

Results at finite T will get the light soon, lets go to the case of finite densities and zero T

Pressure for cold and dense matter

order δ^0

$$P_{0,f}^{RGOPT}(\mu) = \frac{N_c}{12\pi^2} \left[\mu p_F \left(\mu^2 - \frac{5}{2} m^2 \right) + \frac{3}{2} m^4 \left(L_\mu - \frac{3}{4} \right) \right] + N_c \frac{m^4}{(4\pi)^2 b_0 g},$$

where $p_F = \sqrt{\mu^2 - m^2}$ and $L_\mu \equiv \ln[(\mu + p_F)/M]$.

$$f_{\text{PMS}} = \frac{\partial P^{RGOPT}}{\partial m} \equiv 0.$$

$$\Downarrow$$
$$\overline{m}^2 = \mu^2 \left(\frac{\sqrt{1+4c(\overline{m},\mu,g)}-1}{2c(\overline{m},\mu,g)} \right)$$

$$c(m, \mu, g) = \left(\frac{1}{2b_0g} - \frac{1}{2} + L_\mu \right)^2$$

$$g_{1L}(M) = \frac{g(M_0)}{1 + 2g(M_0) b_0 \ln \left(\frac{M}{M_0} \right)}.$$

$$\begin{aligned}
 P_{1,f}^{RGOPT}(\mu) &= P_{0,f}^{RGOPT}(\mu) - N_c \frac{m^4}{(4\pi)^2} \left(\frac{\gamma_0}{b_0} \right) \left(\frac{1}{b_0 g} \right) + m^4 \left(2 \frac{\gamma_0}{b_0} - 1 \right) s_1 \\
 &+ N_c \frac{m^2}{8\pi^2} \left(\frac{\gamma_0}{b_0} \right) \left[m^2 (1 - 2L_\mu) + 2\mu p_F \right] \\
 &- \frac{gd_A}{4(2\pi)^4} \left[m^4 \left(\frac{1}{4} - 4L_\mu + 3L_\mu^2 \right) + \mu^2 (\mu^2 + m^2) + m^2 \mu p_F (4 - 6L_\mu) \right]
 \end{aligned}$$

sols for $\mu = 0$

$$\ln \left(\frac{\bar{m}}{M} \right) \approx 0.0284258 \pm 1.22604i,$$

$$\bar{g} \approx 6.74555 \pm 3.19202i \quad (\bar{\alpha}_s = 0.536794 \pm 0.254013i).$$

Problem!!!

f_{PMS} only give real solutions of L_m if $g \leq 2.63799$ or $\alpha_s \leq 0.209924$

This is not a new problem [Kneur and Neveu, PhysRevD.88.074025 (2013)]

Renormalization scheme change (RSC)

How to recover real solutions?

$m \rightarrow m(1 + B_1 g + B_2 g^2 + \dots)$, (this before applying the RGOPT prescription n)

One must use as few new parameters as possible, then $m \rightarrow m(1 + B_2 g^2)$
This add an extra term $-4g m^4 s_0 B_2$ to the pressure

one requires the RSC to give the real \bar{m} solution the *closest* to the original $\overline{\text{MS}}$ -scheme

$$f_{RSC} = \frac{\partial f_{RG}}{\partial g} \frac{\partial f_{PMS}}{\partial m} - \frac{\partial f_{RG}}{\partial m} \frac{\partial f_{PMS}}{\partial g} \equiv 0$$

sols $\mu = 0$

solving $f_{PMS} = 0$, $f_{RG} = 0$ and $f_{RSC} = 0$ simultaneously one gets
 $\ln(\bar{m}/M) \simeq -0.43957$, $\bar{g} = 4.8905$ ($\bar{\alpha}_s = 0.39$) and $\bar{B}_2 = -0.002215$

PROBLEM!!!
Imaginary sols for $\mu \neq 0$

A simpler formalism

$f_{PMS} = 0$ and $f_{RSC} = 0$ ($T = \mu = 0$)

For $M = 1$ GeV

$\overline{B}_2 \simeq -0.00224$, $\ln(\overline{m}/M) \simeq -0.331$.

$$\ln \frac{M}{\Lambda_{\overline{\text{MS}}}} = \frac{1}{2b_0 g} + \frac{b_1}{2b_0^2} \ln \left(\frac{b_0 g}{1 + \frac{b_1}{b_0} g} \right)$$

Real results at $\mu \neq 0!!!$

Condensate at $T = \mu = 0$

RGOPT order δ^0

$$\frac{\langle \bar{q}q \rangle_{\delta^0}}{N_c}(m, 0, 0) = \frac{m^3}{2\pi^2} \left[\frac{1}{2} - \ln \left(\frac{m}{\Lambda} \right) \right],$$

$$\frac{\langle \bar{q}q \rangle_{\delta^0}}{N_c}(\bar{m}_{\delta^0}, 0, 0) = \frac{\Lambda_{\overline{\text{MS}}}^3 e^{3/2}}{4\pi^2 g b_0}.$$

RGOPT order δ

$$\begin{aligned} \frac{\langle \bar{q}q \rangle_{\delta^1}}{N_c} &= \frac{m^3}{2\pi^2} \left\{ \frac{1}{2} - \ln \left(\frac{m}{\Lambda} \right) - \frac{3\gamma_0}{2b_0} \left[\frac{1}{6} - \ln \left(\frac{m}{\Lambda} \right) \right] \right. \\ &\quad \left. + 2g(N_c^2 - 1) \left(\ln^2 \left(\frac{m}{\Lambda} \right) - \frac{5}{6} \ln \left(\frac{m}{\Lambda} \right) + \frac{5}{12} \right) \right\}. \end{aligned}$$

$$\frac{\langle \bar{q}q \rangle_{\delta^1}}{N_c} \approx 1.31478 \Lambda^3.$$

[Kneur and Neveu (2015)] $\rightarrow \lim_{m \rightarrow 0} \langle \bar{q}q \rangle = \pi \rho$ Banks–Casher relation.

Results at $\mu \neq 0$

[Kneur, Pinto, Restrepo, arXiv:1908.08363v1]

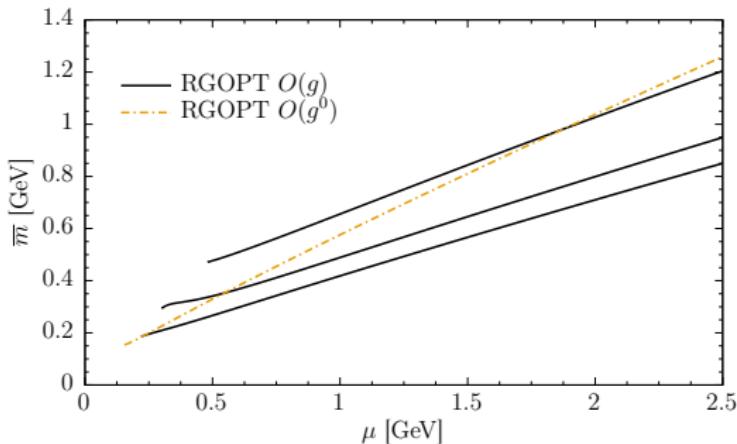


Figure: The optimized mass as a function of the chemical potential at $M = \mu, 2\mu$ and 4μ at order- g^0 (dot-dashed) and order- g (continuous). For the latter the upper curve corresponds to $M = \mu$, the central curve to $M = 2\mu$, and the lower curve to $M = 4\mu$.

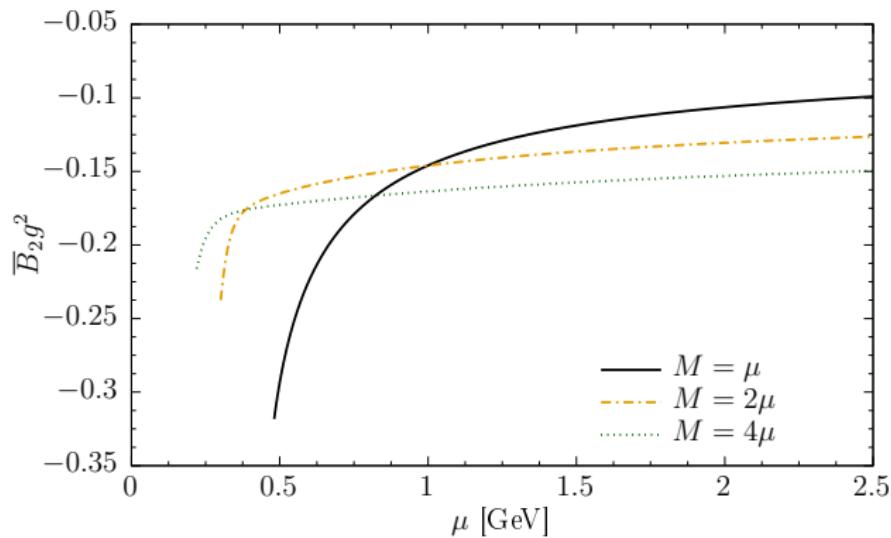


Figure: The optimized RSC $\bar{B}_2(\mu)g^2(\mu)$ quantity as a function of the chemical potential at $M = \mu, 2\mu$ and 4μ at order- g .

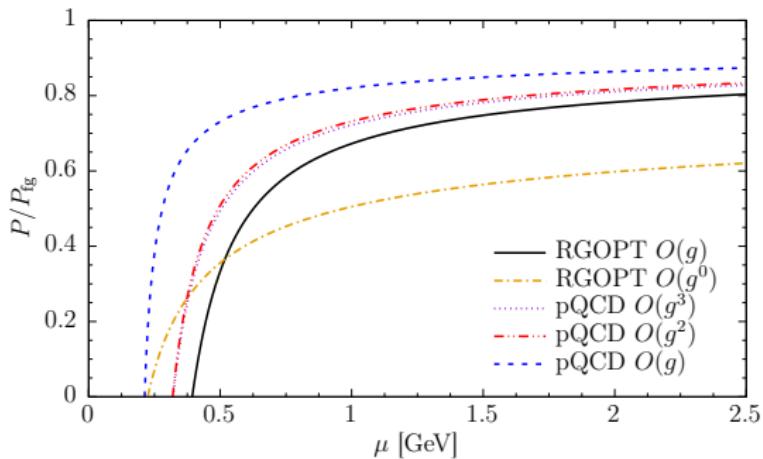


Figure: The normalized pressure as a function of the chemical potential at the central scale $M = 2\mu$. pQCD results for orders g , g^2 , and g^3 (LL term) are compared with the RGOPT results at orders g^0 (one loop) and g (two loop).

$$\frac{P_{pQCD}}{P_{fg}} = 1 - \frac{2}{\pi} \alpha_s(M) - \alpha_s^2(M) \left\{ 0.303964 \ln \alpha_s(M) + \left[0.874355 + 0.911891 \ln \left(\frac{M}{\mu} \right) \right] \right\}$$

$$- 0.266075 \alpha_s^3(M) \ln^2 \alpha_s$$

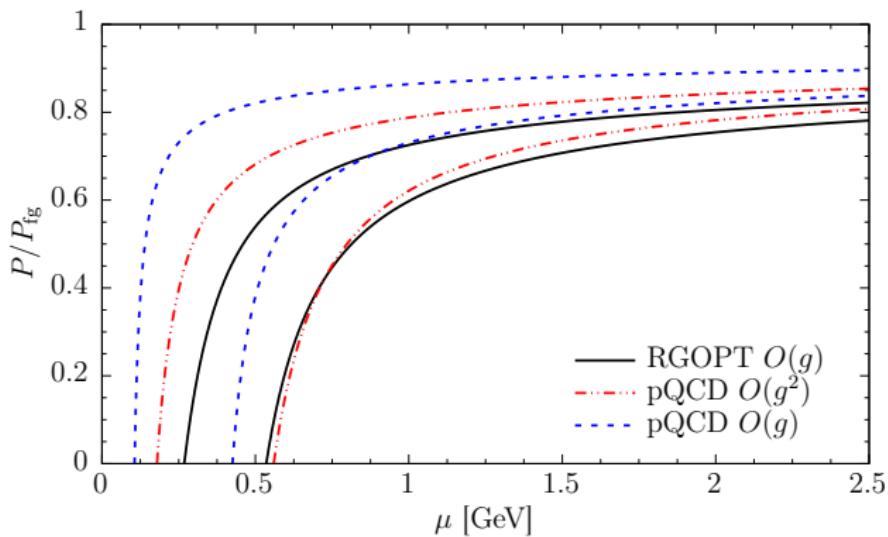


Figure: The normalized pressure as a function of the chemical potential. pQCD results for orders g , g^2 , and g^3 are compared with RGOPT at orders g^0 (one loop) and g (two loop). In each case the upper curve corresponds to $M = 4\mu$, the central curve to $M = 2\mu$, and the lower curve to $M = \mu$.

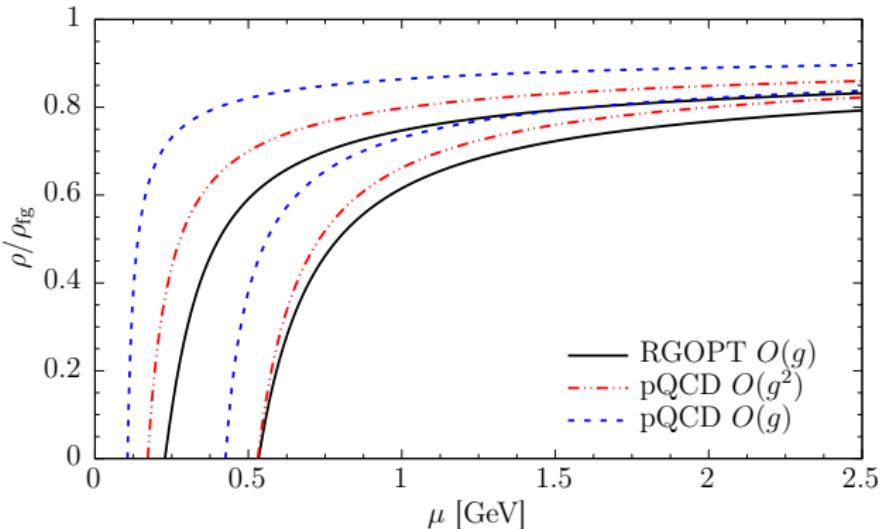


Figure: The quark number density as a function of the chemical potential at $M = \mu$, 2μ and 4μ . pQCD results for orders g , and g^2 are compared with the RGOPT at orders g^0 (one loop) and g (two loop). In each case the upper curve corresponds to $M = 4\mu$, the central curve to $M = 2\mu$, and the lower curve to $M = \mu$.

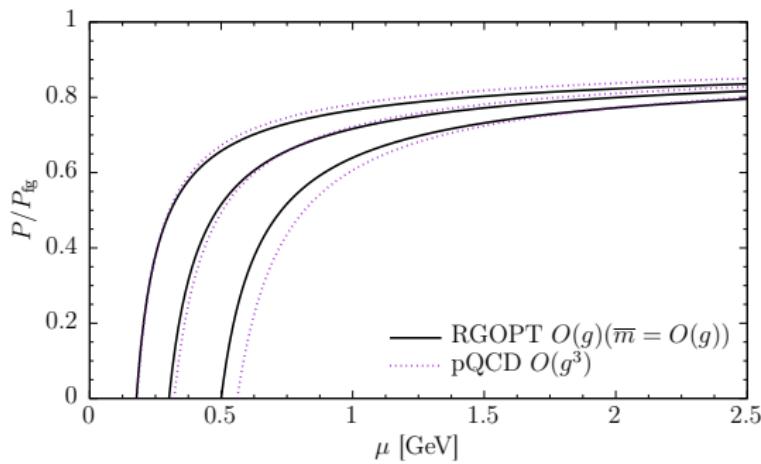
A simpler alternative

- Solve the complete RG equation

$$\left[M \frac{\partial}{\partial M} + \beta \frac{\partial}{\partial g} - \gamma_m m \frac{\partial}{\partial m} \right] P_{1,f}^{\text{RGOPT}} = 0 \rightarrow \bar{m}$$

- Expand up to order $g \rightarrow \bar{m}^2 = \frac{9}{7\pi^2} \left(1 - \frac{\sqrt{43}}{8}\right) g \mu^2 + \mathcal{O}(g^2)$

- Finally replace in the 2L RGOPT pressure, $P_{1,f}^{\text{RGOPT}}(\mu)$ (there is not B_2).



Conclusions and perspectives

- RGOPT gives a perturbative evaluation to generate non-perturbative and scale invariant results at “trivial” g^0 order.
- At order g RGOPT reduces the scale dependence in comparison with pQCD.
- Our two loop order is in better agreement with the pQCD order α_s^3 result.
- Next: include gluons (technically very difficult), finite current masses, apply the EoS to describe neutron stars, etc.
- At higher orders RGOPT may present issues as complex solutions and residual scale dependence.

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