Clusters as physical objects and carriers of topological charge and vorticity

João C. Pinto Barros

Albert Einstein Center for Fundamental Physics
Institute for Theoretical Physics, University of Bern

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In collaboration with:

Wolfgang Bietenholz (Mexico City), Stephan Caspar (Seattle) , Manes Hornung, Uwe-Jens Wiese (Bern)

Clusters as physical objects and carriers of topological charge and vorticity

What are clusters?

Objects defined in the path integral of a Quantum Field Theory.

What are they good for?

They are non-perturbative, are physical and offer insight into dynamics of the model

In this talk we will focus on the 1-d O(2) and 2-d O(3) while also referring to 2-d O(2) and 3-d O(4) models.

In these models clusters can carry **topological charge or vorticity.**

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Outline

- **The 1-d O(2) model as an illustrative example**
  - Path integral and instantons
  - Clusters, merons and improved estimators
  - Cluster size distribution

- **The 2-d O(3) model**
  - Instantons, merons and higher charges
  - Clusters as fractal physical objects
  - The divergence of the topological susceptibility: a lesson from clusters

- **The 2-d O(2) and 3-d O(4) models**
  - Fractal dimension and critical exponents
  - BKT transition and vorticity in the 2-d O(2) model

The 1-d O(2) model

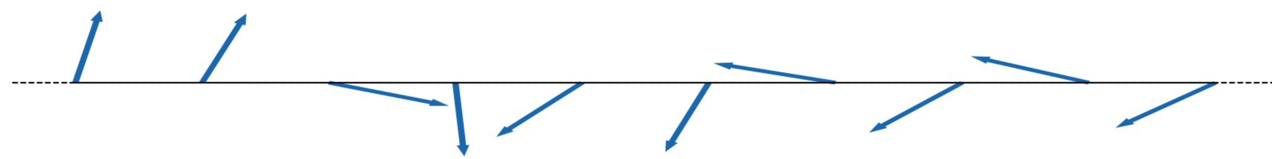
Euclidean action

- Fields are 2 components unit real vectors

$$S\left[\vec{e}\right] = \frac{1}{2g^2}\int_0^\beta d\tau \dot{\vec{e}}\left(\tau\right)^2$$

$$\vec{e}\cdot\vec{e} = 1 \Leftrightarrow \vec{e} = \left(\cos\varphi, \sin\varphi\right)$$

$$S\left[\varphi\right] = \frac{1}{2g^2}\int_0^\beta d\tau \dot{\varphi}\left(\tau\right)^2$$



- Action is invariant under a global rotation

$$\vec{e}\,'\left(\tau\right) = \Lambda\vec{e}\left(\tau\right),\ \Lambda \in \mathrm{O}\left(2\right)$$

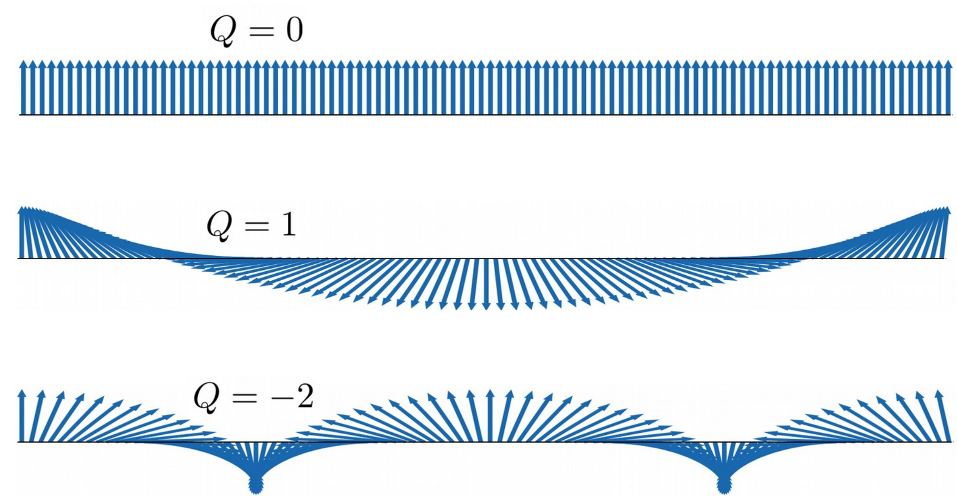
(equivalently $\varphi'\left(\tau\right) = \varphi\left(\tau\right) + \Delta$)

- Equations of motion and classical solutions

$$\ddot{\varphi}\left(\tau\right) = 0 \Leftrightarrow \varphi\left(\tau\right) = \omega\tau + \varphi_0$$

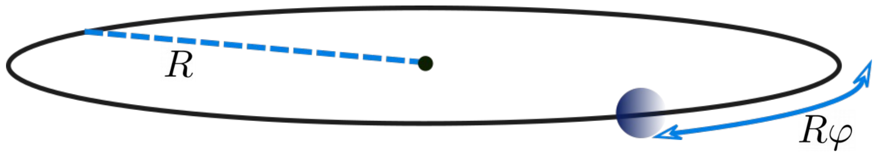
With periodic boundary conditions

$$\varphi_Q\left(\tau\right) = \frac{2\pi}{\beta}Q\tau,\ Q \in \mathbb{Z}$$



The 1-d O(2) model as a particle in a ring

Particle in a ring



$$\hat{H} = \frac{1}{2m}\hat{p}^2 = -\frac{1}{2mR^2}\partial_\varphi^2$$

$$Z = \mathrm{Tr}\left(e^{-\beta H}\right) \to \int\limits_{\varphi(0)=\varphi(\beta)} D\varphi e^{-\frac{1}{2mR^2}\int_0^\beta d\tau \dot{\varphi}(\tau)^2}$$

When comparing this to the 1-d O(2) action we can associate the coupling to the particle in a ring problem

$$S[\varphi] = \frac{1}{2g^2}\int_0^\beta d\tau \dot{\varphi}(\tau)^2 \longrightarrow \boxed{g^2 = mR^2}$$

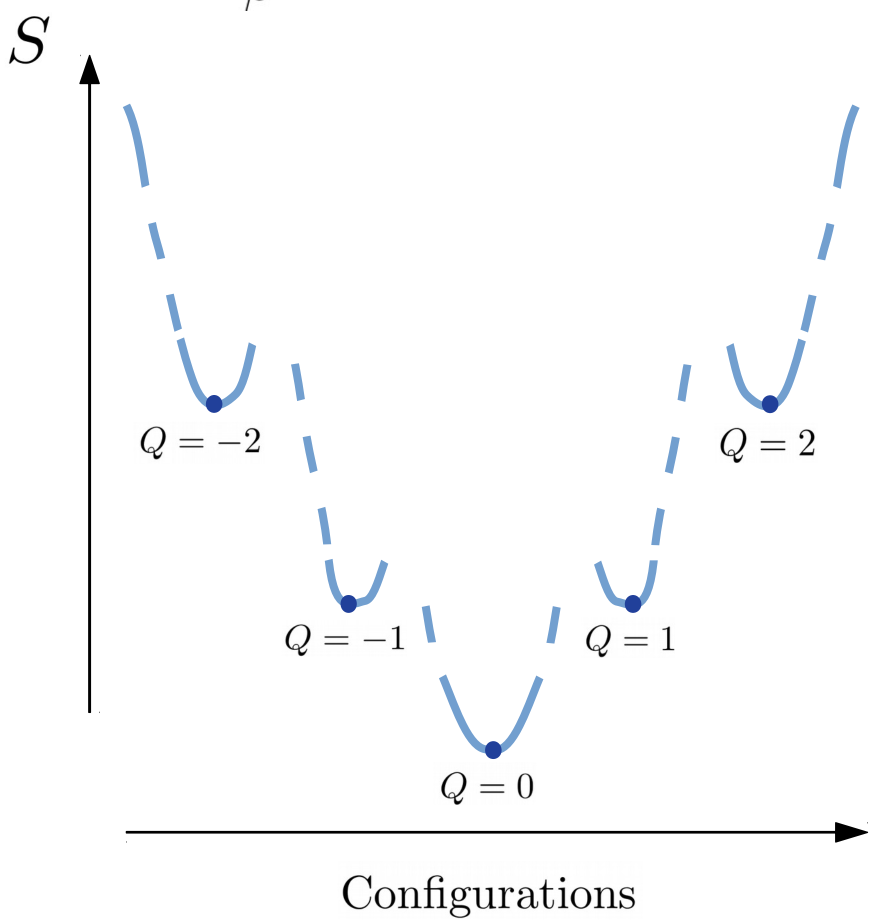
Topology in the 1-d O(2) model

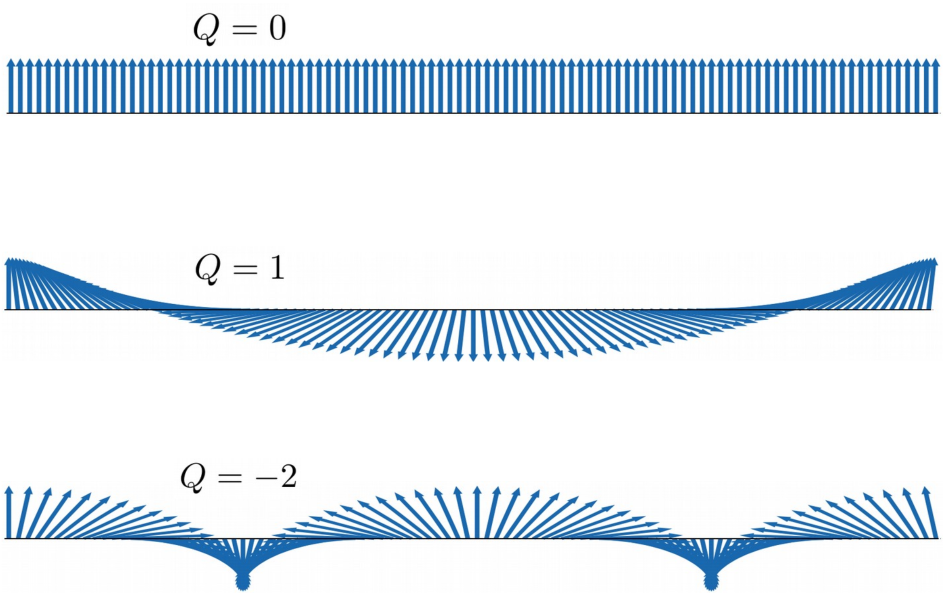
A closer look at the classical solution

$$\varphi_Q(\tau) = \frac{2\pi}{\beta} Q\tau, \; Q \in \mathbb{Z}$$

Action of the solution

$$S_Q = \frac{4\pi^2 Q^2}{2g^2\beta}$$

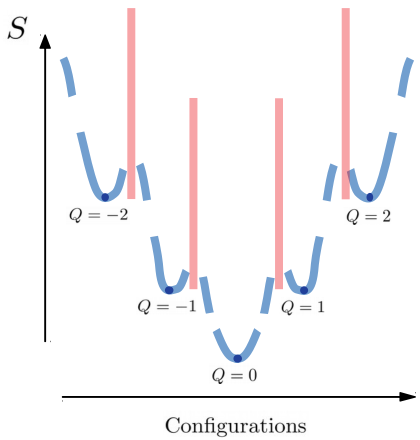


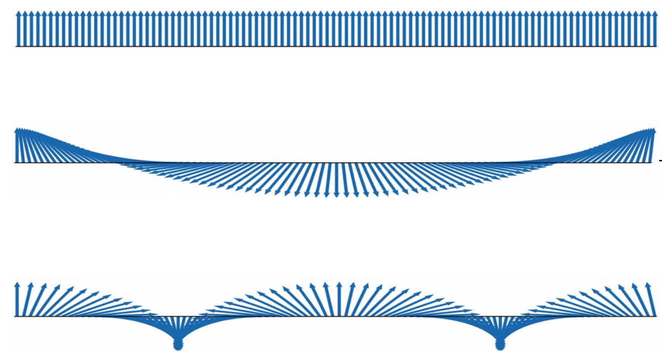


Instantons

Topology in the 1-d O(2) model

Infinite action barriers separate instaton solutions. Q is the **topological charge** of a given configuration.

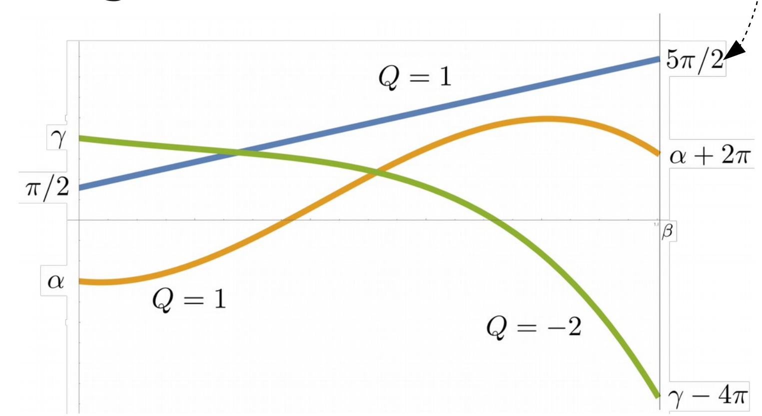




Other configurations, which are not classical solutions, can be classified according to their topological charge.

$$Q=\frac{1}{2\pi}\int_{0}^{\beta} d\tau\,\dot{\varphi}(\tau)=\frac{\varphi(\beta)-\varphi(0)}{2\pi}$$

2π times an integer due to periodic boundary conditions



The 1-d O(2) model on a lattice

Different lattice discretizations can lead to the same continuum limit:

$$S\left[\vec{e}\right] = \frac{1}{2g^2}\int_0^\beta d\tau \dot{\vec{e}}\left(\tau\right)^2$$

- Consider a lattice with N sites with nearest neighbor interactions

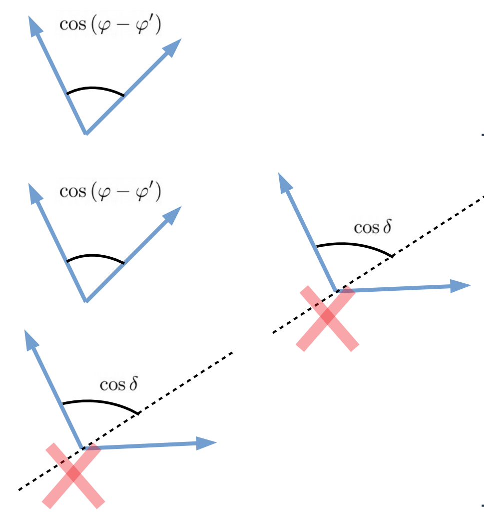
$$S\left[\vec{e}\right] = \sum_{\langle i,j\rangle} s\left(\vec{e}_i, \vec{e}_j\right) \longrightarrow Z = \sum_{\{\vec{e}\}} \prod_{\langle i,j\rangle} e^{-s\left[\vec{e}_i, \vec{e}_j\right]}$$

- Possible choices for the action

$$s_{\text{standard}}\left(\vec{e}, \vec{e}\,'\right) = \frac{1}{g_a^2}\left(1 - \vec{e}\cdot\vec{e}\,'\right)$$

$$s_{\text{constraint}}\left(\vec{e}, \vec{e}\,'\right) = \begin{cases} \frac{1}{g_a^2}\left(1 - \vec{e}\cdot\vec{e}\,'\right) & \text{if } \ \vec{e}\cdot\vec{e}\,' > \cos\delta \\ +\infty & \text{otherwise} \end{cases}$$

$$s_{\text{topological}}\left(\vec{e}, \vec{e}\,'\right) = \begin{cases} 0 & \text{if } \ \vec{e}\cdot\vec{e}\,' > \cos\delta \\ +\infty & \text{otherwise} \end{cases}$$



Used for

$2 - d\ \mathrm{O}\left(2\right)$

$2 - d\ \mathrm{O}\left(3\right)$

$3 - d\ \mathrm{O}\left(4\right)$

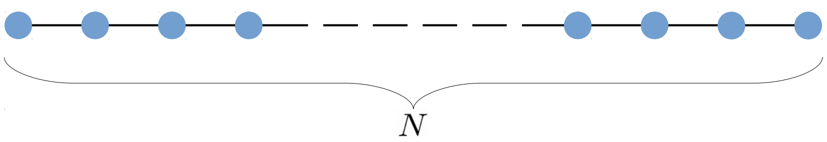
$$e^{-s_{\text{Villain}}\left[\varphi, \varphi'\right]} = \sum_{n\in\mathbb{Z}} e^{-\frac{1}{g_a^2}\sum_{n=0}^{N-1}\left(\varphi - \varphi' + 2\pi n\right)^2}$$

$1 - d\ \mathrm{O}\left(2\right)$

Lattice 1-d O(2) and the continuum limit

Taking the naive continuum limit for all but the topological action

$$e^{-s_{\mathrm{Villain}}[\varphi,\varphi']} = \sum_{n\in\mathbb{Z}} e^{-\frac{1}{g_a^2}\sum_{n=0}^{N-1}(\varphi-\varphi'+2\pi n)^2}$$



$$s_{\mathrm{standard}}\left(\vec{e},\vec{e}\,'\right) = \frac{1}{g_a^2}\left(1-\vec{e}\cdot\vec{e}\,'\right)$$

$$s_{\mathrm{constraint}}\left(\vec{e},\vec{e}\,'\right) = \begin{cases} \frac{1}{g_a^2}\left(1-\vec{e}\cdot\vec{e}\,'\right) & \text{if } \vec{e}\cdot\vec{e}\,' < \cos\delta \\ +\infty & \text{otherwise} \end{cases}$$

Continuum limit

$$\boxed{\begin{array}{l} g_a \to 0 \\ N \to +\infty \end{array}}$$

$$s_{\mathrm{topological}}\left(\vec{e},\vec{e}\,'\right) = \begin{cases} 0 & \text{if } \vec{e}\cdot\vec{e}\,' < \cos\delta \\ +\infty & \text{otherwise} \end{cases}$$

W. Bietenholz, U. Gerber, M. Pepe, UJW, JHEP (2010) 1012:020

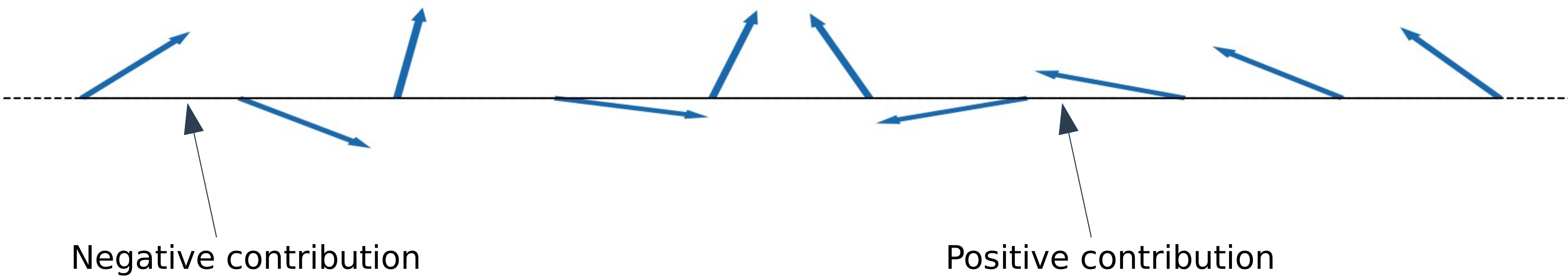
$$\frac{1}{g_a^2}\sum_{i=0}^{N-1}\left(1-\vec{e}_i\cdot\vec{e}_{i+1}\right) = \frac{1}{2g_a^2}\sum_{i=0}^{N-1}\left(\vec{e}_i-\vec{e}_{i+1}\right)^2 = \frac{a^2}{2g_a^2}\sum_{i=0}^{N-1}\frac{\left(\vec{e}_i-\vec{e}_{i+1}\right)^2}{a^2} \simeq \frac{a}{2g_a^2}\int_0^{(N-1)a} d\tau\,\dot{\vec{e}}(\tau)^2$$

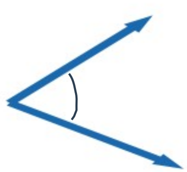
$$\boxed{\begin{array}{l} g_a a^{-1} \to mR^2 \\ Na \to \beta \end{array}}$$

Topology on the lattice 1-d O(2) model

How to extend the concept of topological charge to the lattice discretization?

$$Q = \frac{1}{2\pi} \int_0^\beta d\tau \dot{\varphi}(\tau)$$

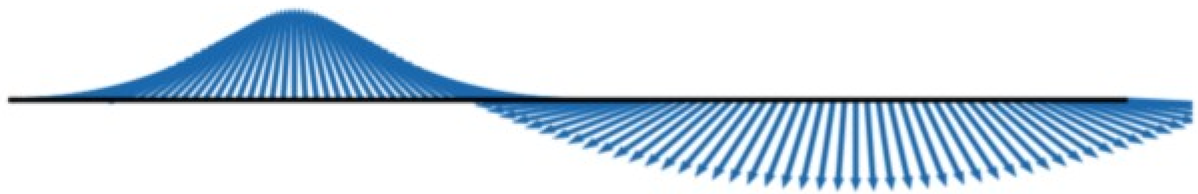




$$Q = \frac{1}{2\pi} \sum_{i=0}^{N-1} (\varphi_{i+1} - \varphi_i \mod 2\pi)$$

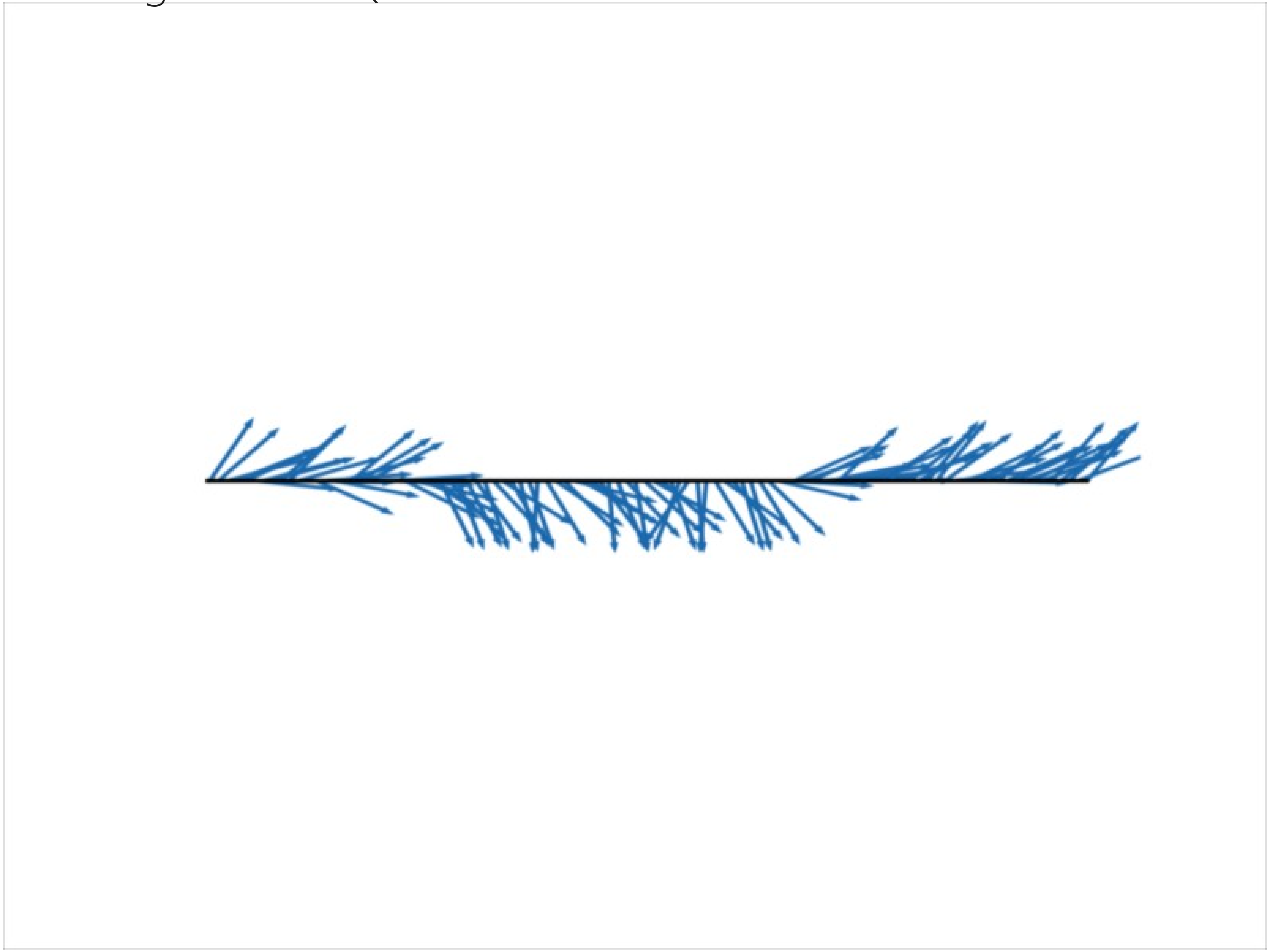
Topology on the lattice 1-d O(2) model

Instantons Q=1



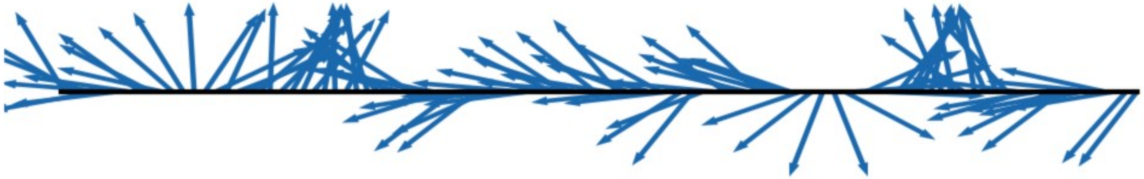
Topology on the lattice 1-d O(2) model

Random configuration Q=0



Topology on the lattice 1-d O(2) model

Random configuration Q=1



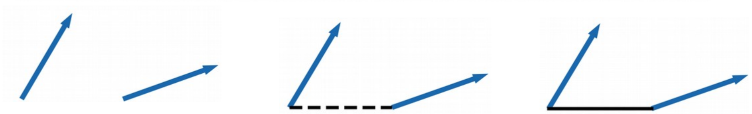
Wolff cluster algorithm

The weight of a configuration is the product of pairwise interactions

$$Z = \sum_{\{\vec{e}\}} \prod_{\langle i,j \rangle} e^{-s[\vec{e}_i,\vec{e}_j]}$$

We introduce **bond variables**, that, when integrated out, give rise to the original model

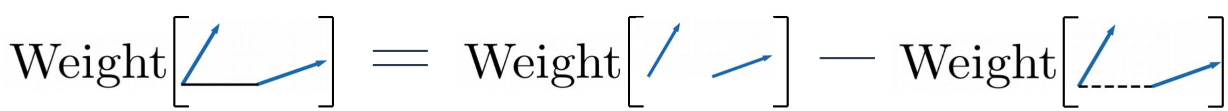
$$e^{-s[\vec{e}_i,\vec{e}_j]} = w_0\left(\vec{e}_i, \vec{e}_j\right) + w_1\left(\vec{e}_i, \vec{e}_j\right)$$



Bond **is not** put   Bond **is** put

Weight for not putting a bond is the weight of having the spins in opposite side of a given line (consider, e.g., the x axis)

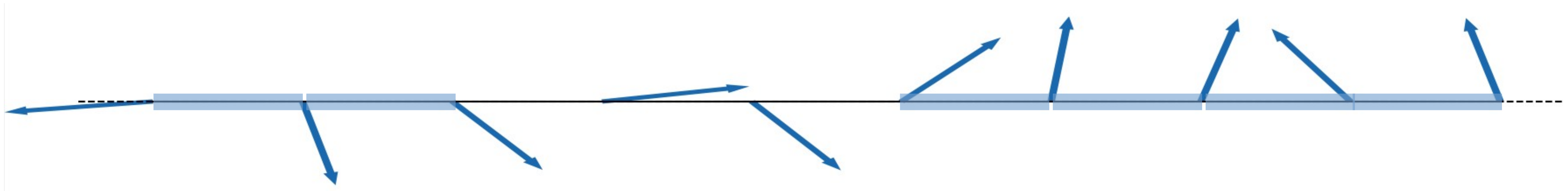
The remaining weight (if any) is assigned to the configuration with a bond



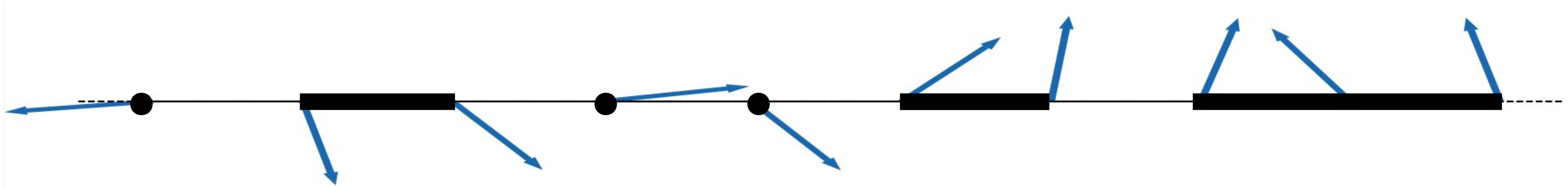
U. Wolff, Phys. Rev. Lett. 62 (1989) 361; Nucl. Phys. B334 (1990) 581

Wolff cluster algorithm: example

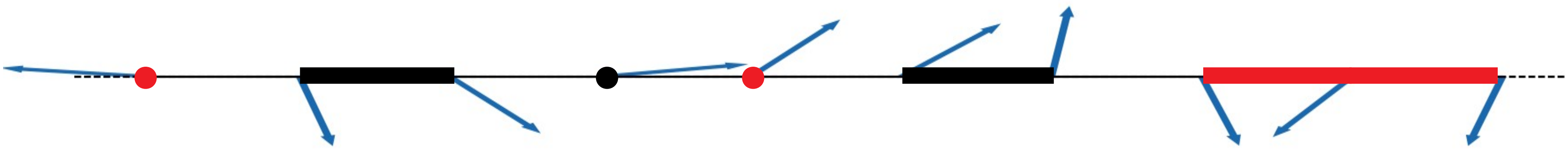
Only bonds on the same side of the axis **can** be put



However, they are not necessarily put



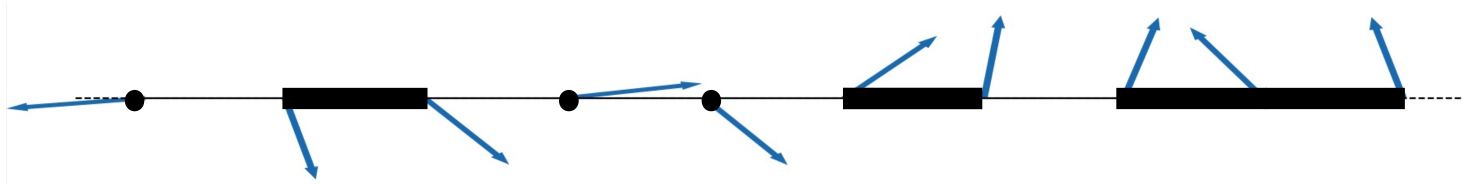
Clusters are set of spins that are connected through bonds. Spins not connected by bonds are *single site clusters*. Clusters are flipped with probability 1/2

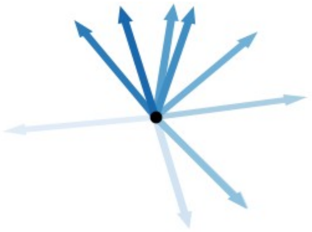


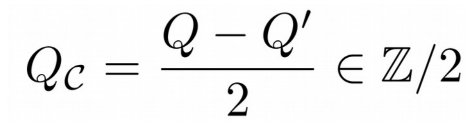
U. Wolff, Phys. Rev. Lett. 62 (1989) 361; Nucl. Phys. B334 (1990) 581

Cluster charges

We can calculate the cluster charge of any cluster starting from any configuration



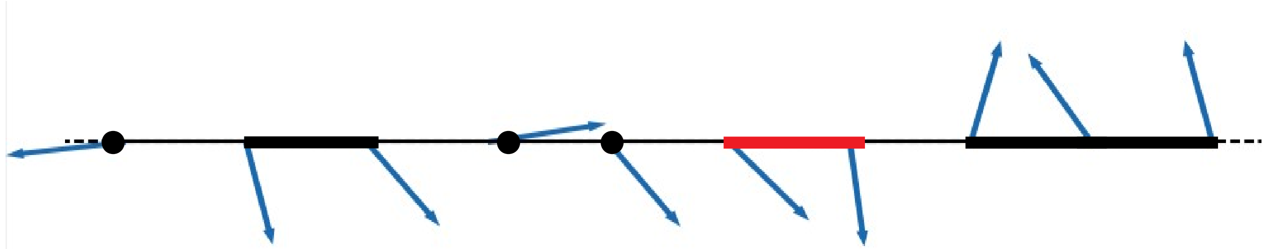


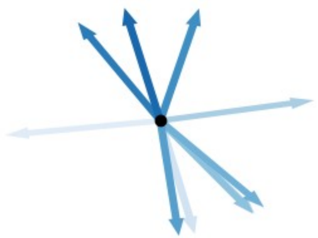


Once we flip a cluster, we can define its charges as i.e., through the effect it has on the configuration

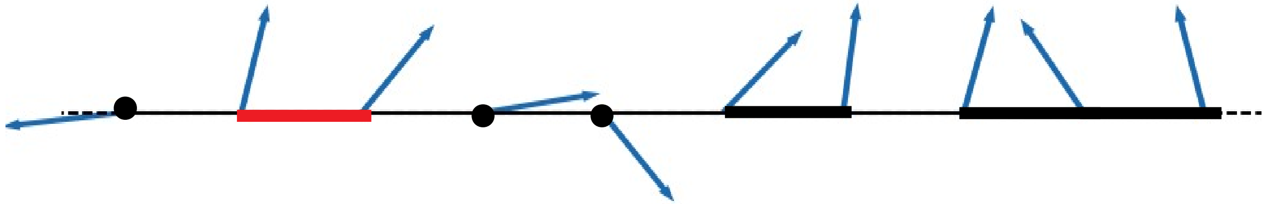
$$Q_C = \frac{Q - Q'}{2} \in \mathbb{Z}/2$$

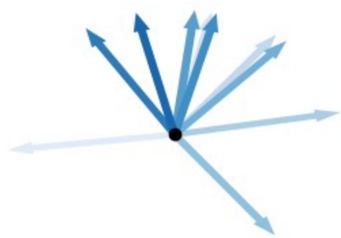
W Bietenholz, A Pochinsky, UJ Wiese, Phys. Rev. Letters 75 (24), 4524





- Cluster does not change the charge of the configuration: neutral cluster

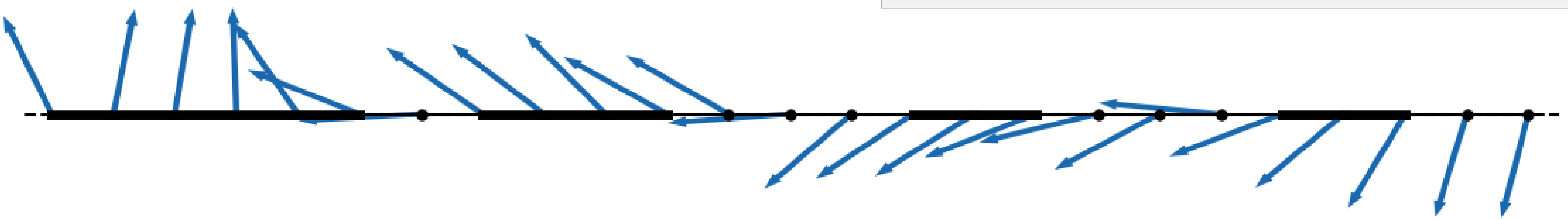




- Cluster changes the charge of the configuration by 1: meron cluster

Wolff Clusters overview

Example of a cluster decomposition for 25 sites.
There are 13 clusters.



A spin configuration with $n_{\mathcal{C}}$ clusters can be regarded as an element of a sub-ensemble of $2^{n_{\mathcal{C}}}$ configurations, all with the **same weight**.

Cluster charges are independent of the orientation of all others.

A flipped cluster carries opposite charge to the original one.

$$Q = \sum_{\mathcal{C}} Q_{\mathcal{C}}$$

The total charge of a configuration is the sum of the individual charges of the clusters

Improved estimators

Cluster are also directed related to observables

- Ex: Topological Susceptibility

$$\chi_t = \frac{\langle Q^2 \rangle - \langle Q \rangle^2}{V} = \frac{\langle Q^2 \rangle}{N} = \frac{1}{N} \left\langle \sum_{\mathcal{C},\mathcal{C}'} Q_{\mathcal{C}} Q_{\mathcal{C}'} \right\rangle = \frac{1}{N} \left\langle \sum_{\mathcal{C}} Q_{\mathcal{C}}^2 \right\rangle$$

In this model clusters have charge zero (neutral) or charge +/- 1/2 (**merons**)

$$= \frac{1}{4N} \langle N_{\mathrm{M}} \rangle = \frac{1}{4N} \sum_{|\mathcal{C}|} n_{\mathrm{M}} \left(|\mathcal{C}| \right)$$

If this distribution exists in the continuum limit

$$\rightarrow \frac{1}{4V} \int_0^V ds \rho_{\mathrm{M}} \left(s \right)$$

Does such distribution exist?

The cluster size distribution

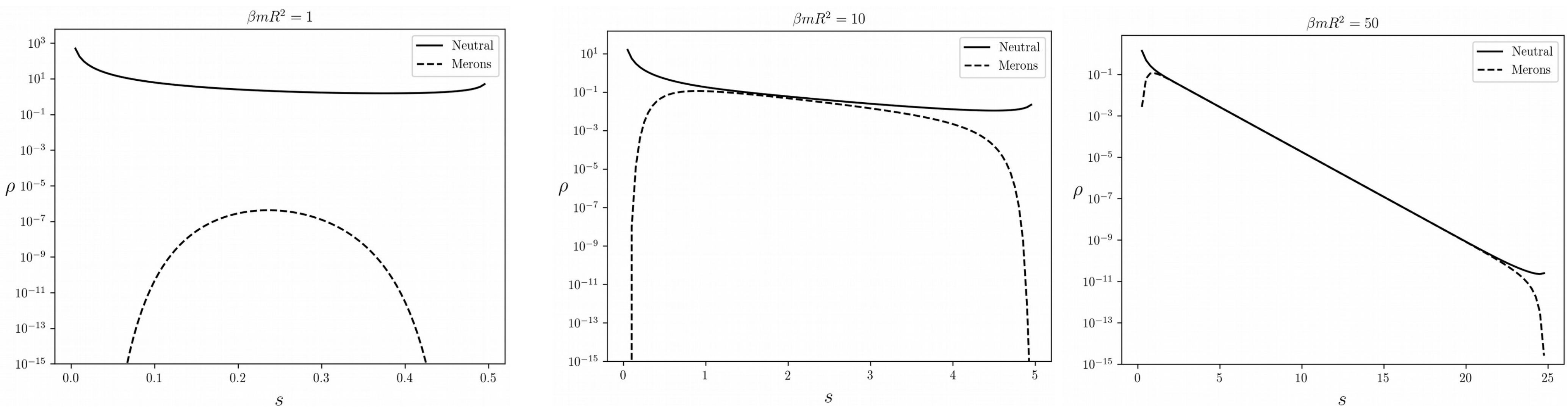
It is possible to compute the cluster size distribution for the Villain action, analytically, for finite lattice spacing and take the continuum limit.

$$e^{-s[\vec{e}_i,\vec{e}_j]} = w\left(\vec{e}_i,\vec{e}_j\right) = w_0\left(\vec{e}_i,\vec{e}_j\right) + w_1\left(\vec{e}_i,\vec{e}_j\right)$$

$$Z = \sum_{\{\vec{e}\}} \prod_{\langle i,j \rangle} e^{-s[\vec{e}_i,\vec{e}_j]} = \mathrm{Tr}\left(w^N\right)$$

$$f_{|\mathcal{C}|} = N \frac{\mathrm{Tr}\left(w_0 w_1^{|\mathcal{C}|-1} w_0 w^{N-|\mathcal{C}|-1}\right)}{Z}, \quad |\mathcal{C}| < N$$

$$f_N = \frac{N \mathrm{Tr}\left(w_0 w_1^{N-1}\right) + \mathrm{Tr}\left(w_1^N\right)}{Z}$$



An overview over the 1-d O(2) model here

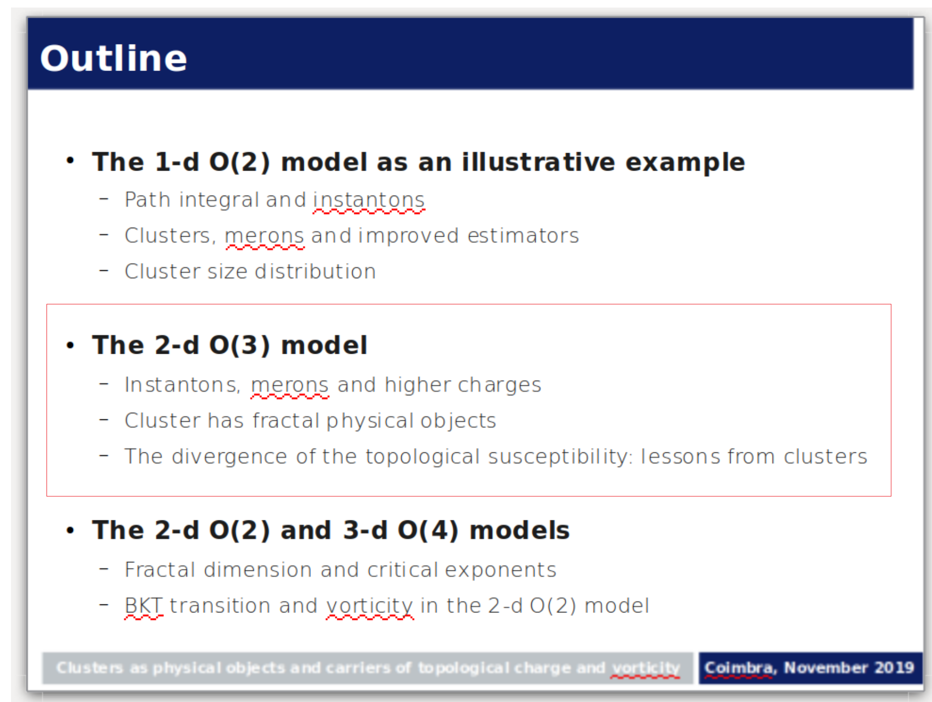
Introduce important concepts:

- Topological charge and instantons.
- Clusters and cluster algorithms

Proof of principle:

- Clusters exist in the continuum limit and **are physical.**

We want now to extend this to the 2-d O(3) model


Outline

- **The 1-d O(2) model as an illustrative example**
  - Path integral and instantons
  - Clusters, merons and improved estimators
  - Cluster size distribution
- **The 2-d O(3) model**
  - Instantons, merons and higher charges
  - Cluster has fractal physical objects
  - The divergence of the topological susceptibility: lessons from clusters
- **The 2-d O(2) and 3-d O(4) models**
  - Fractal dimension and critical exponents
  - BKT transition and vorticity in the 2-d O(2) model

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The 2-d O(3) model

Low-energy effective field theories for spin chains

It shares interesting features with QCD

- Asymptotically free
- Dynamically generated mass gap
- Confinement

  (It is the same model as CP(1), where CP(N-1) is a family of models which share all the above features)
- Non trivial topology
  - Is the topology of the model fundamentally flawed?

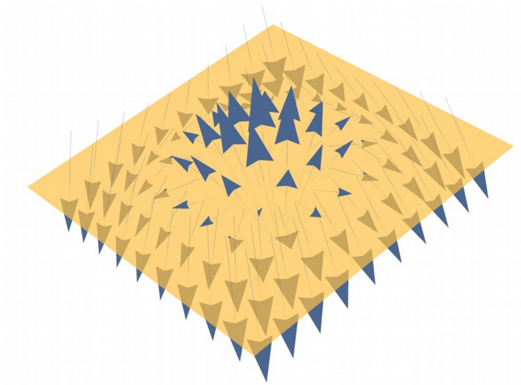
The 2-d O(3) model

Euclidean action

$$S[\vec{e}] = \frac{1}{2g^2} \int d^2x \, \partial_\mu \vec{e} \cdot \partial_\mu \vec{e}$$

Topological charge

$$Q[\vec{e}] = \frac{1}{8\pi} \int d^2x \, \varepsilon_{\mu\nu} \vec{e} \cdot (\partial_\mu \vec{e} \times \partial_\nu \vec{e})$$



On the lattice

$$S\left[\vec{e}\right] = \sum_{\langle i,j \rangle} s\left(\vec{e}_i, \vec{e}_j\right)$$

$$s_{\text{constraint}}\left(\vec{e}, \vec{e}\,'\right) = \begin{cases} \frac{1}{g_a^2}\left(1 - \vec{e} \cdot \vec{e}\,'\right) & \text{if } \vec{e} \cdot \vec{e}\,' > \cos\delta \\ +\infty & \text{otherwise} \end{cases}$$

$$s_{\text{topological}}\left(\vec{e}, \vec{e}\,'\right) = \begin{cases} 0 & \text{if } \vec{e} \cdot \vec{e}\,' > \cos\delta \\ +\infty & \text{otherwise} \end{cases}$$

Constraint angle is essential to guarantee that each cluster has well defined topological charge.

Topology on the 2-d O(3) model

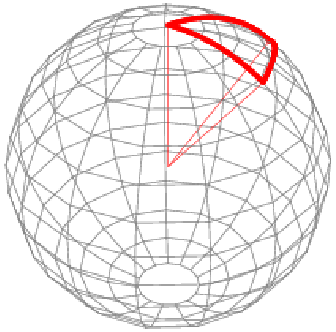
Naive discretization of the topological charge does not guarantee integer topological charge at finite lattice spacing

$$Q[\vec{e}] = \frac{1}{8\pi} \int d^2x \; \varepsilon_{\mu\nu} \vec{e} \cdot (\partial_\mu \vec{e} \times \partial_\nu \vec{e})$$

We will use a geometric definition of the topological charge:

- Three spins define an oriented area on the sphere

$$Q[\vec{e}] = \frac{1}{4\pi} \sum_{t_{xyz}} A_{xyz} \in \mathbb{Z}$$

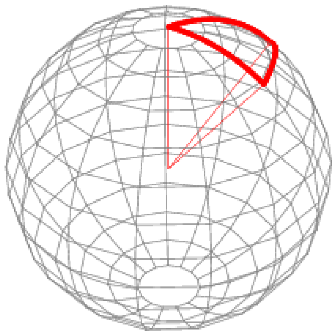


B. Berg and M. Lüscher, Nucl. Phys. B190 (1981) 412
W. Bietenholz, U. Gerber, M. Pepe, U.-J. Wiese, JHEP (2010) 1012:020

Topology on the 2-d O(3) model

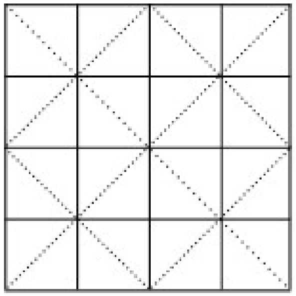
How do we choose these three spins?

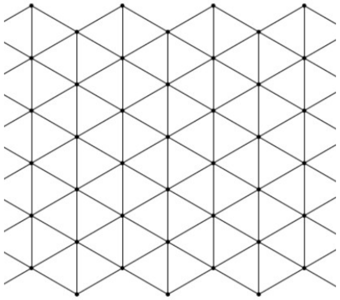
$$Q[\vec{e}] = \frac{1}{4\pi} \sum_{t_{xyz}} A_{xyz} \in \mathbb{Z}$$



We can choose **triangular** lattice or a **triangulated square** lattice

$$s_{\text{constraint}}(\vec{e}, \vec{e}\,') = \begin{cases} \frac{1}{g_a^2}(1 - \vec{e} \cdot \vec{e}\,') & \text{if } \ \vec{e} \cdot \vec{e}\,' > \cos\delta \\ +\infty & \text{otherwise} \end{cases}$$





$$s_{\text{topological}}(\vec{e}, \vec{e}\,') = \begin{cases} 0 & \text{if } \ \vec{e} \cdot \vec{e}\,' > \cos\delta \\ +\infty & \text{otherwise} \end{cases}$$

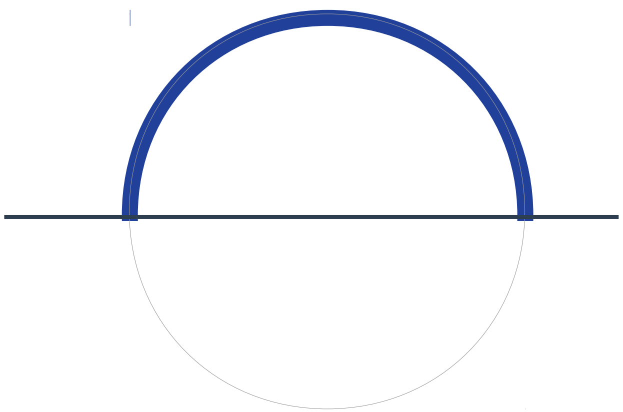
Constraint action in the triangular lattice.

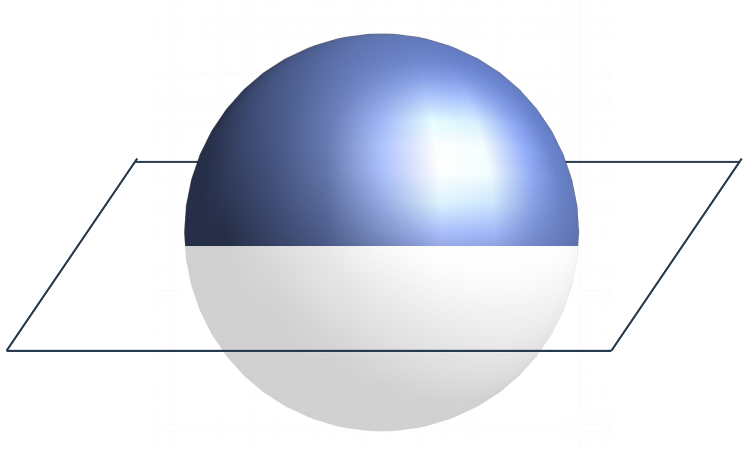
Topological action in the triangulated lattice.

1-d O(2) clusters vs 2-d O(3) clusters

The refection line is now a reflection plane. A bond can only be put if the spins are on the same line of the refection.

Merons now **cover half hemisphere** of a sphere (before the would cover a semi-circle).





Clusters can have **any** integer or half-integer topological charge (before only $\pm 1/2$).

Topology and the Semi-Classical Approx.

Continuum

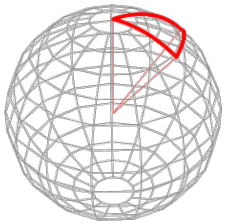
$$S[\vec{e}] = \frac{1}{2g^2}\int d^2x\ \partial_\mu \vec{e}\cdot\partial_\mu\vec{e}$$

$$Q[\vec{e}] = \frac{1}{8\pi}\int d^2x\ \varepsilon_{\mu\nu}\vec{e}\cdot(\partial_\mu\vec{e}\times\partial_\nu\vec{e})$$

Lattice

$$S\left[\vec{e}\right] = \sum_{x,\mu} s_{x\mu}$$

$$Q[\vec{e}] = \frac{1}{4\pi}\sum_{t_{xyz}} A_{xyz} \in \mathbb{Z}$$



Topological susceptibility $\chi_t = \left(\langle Q^2\rangle - \langle Q\rangle^2\right)/V$ Expectation: $\chi_t\xi^2 \to \mathrm{const}$

M. Lüscher, Nucl. Phys. B200 (1982) 61

$$\chi_t \sim \int_0^{+\infty}\frac{d\lambda}{\lambda}$$

Log UV divergence
from small Instantons

$$S_{\mathrm{inst}} = 4\pi\beta$$

$$\chi_t\xi^2 \propto \left(\frac{\xi}{a}\right)^{2-c}$$

Power law UV divergence
from lattice artifacts
--Dislocations--

$$S_{\mathrm{dis}} = 6.69\beta$$

Power law is not observed

Finite topological quantities

Not all topological quantities are ill defined

$$\langle q_0 q_x \rangle \qquad \frac{3\left\langle Q^2\right\rangle^2 - \left\langle Q^4\right\rangle}{\left\langle Q^2\right\rangle} \qquad \chi_t\left(\theta\right) \qquad \left\langle Q\left(\theta\right)\right\rangle$$

J. Balog and M. Niedermaier, Nucl. Phys. B 500 (1997) 421; Phys. Rev. Lett. 78 (1997) 4151

W. Bietenholz, U. Gerber, M. Pepe and U.-J. Wiese, JHEP 1012 (2010) 020

W. Bietenholz, K. Cichy, P. de Forcrand, A. Dromard and U. Gerber, PoS LATTICE2016 (2016) 321

M. B¨ogli, F. Niedermayer, M. Pepe and U.-J. Wiese, JHEP 1204 (2012) 117

Is the topology of the 2d-O(3) model fundamentally flawed

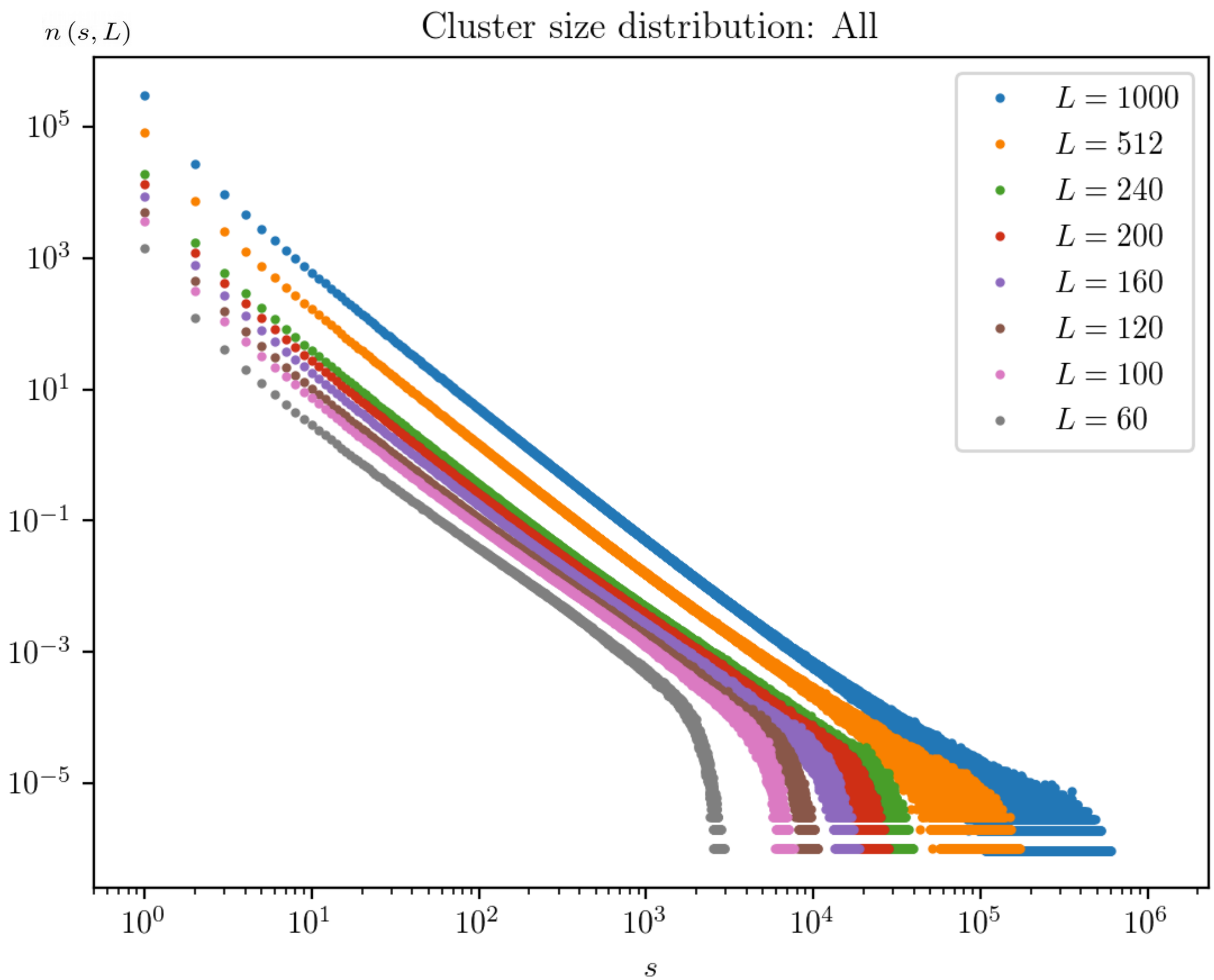
We investigate the cluster size-distribution in order to try to answer this question

- Is the divergence a consequence of a physical cluster size distribution?

$$\chi_t = \frac{\left\langle Q^2\right\rangle}{V} = \frac{1}{V}\sum_s \sum_{q\in\mathbb{Z}} q^2 n_q\left(s\right)$$

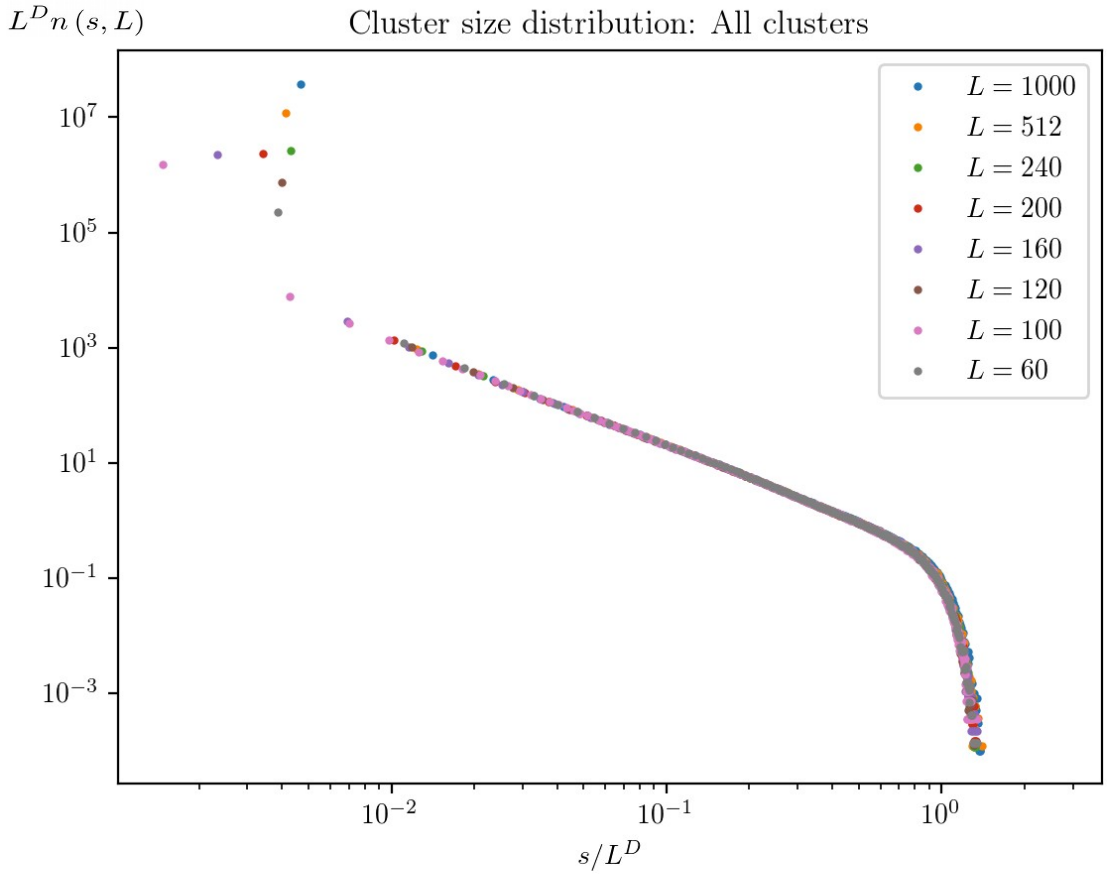
Cluster size distribution measurements

Measurements for the constrained action on a triangular lattice



Cluster size scaling universality

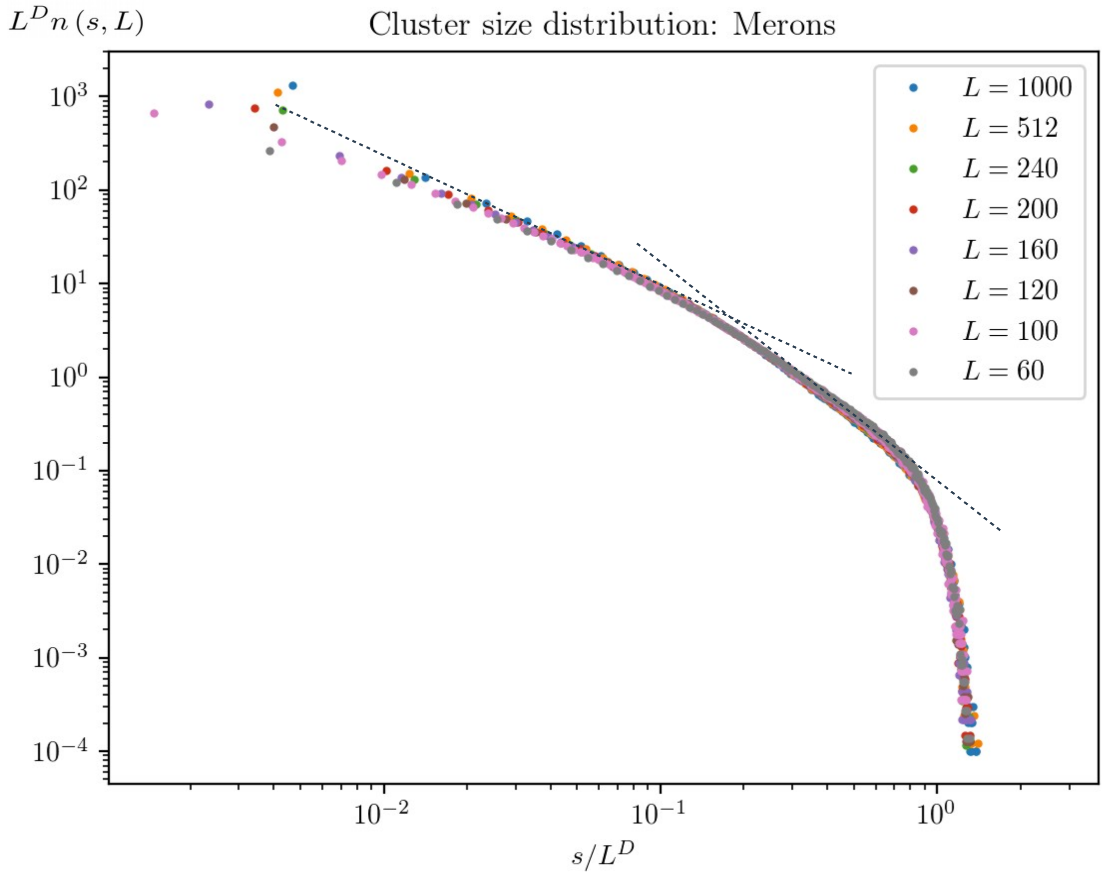
Measurements for the constrained action on a triangular lattice and topological action on the triangulated square lattice



- Clusters scale with fractal dimension $D = 1.88$
- Up to finite size effects and lattice artifacts clusters scale like $1/x^2$

Cluster size scaling universality

Measurements for the constrained action on a triangular lattice and topological action on the triangulated square lattice



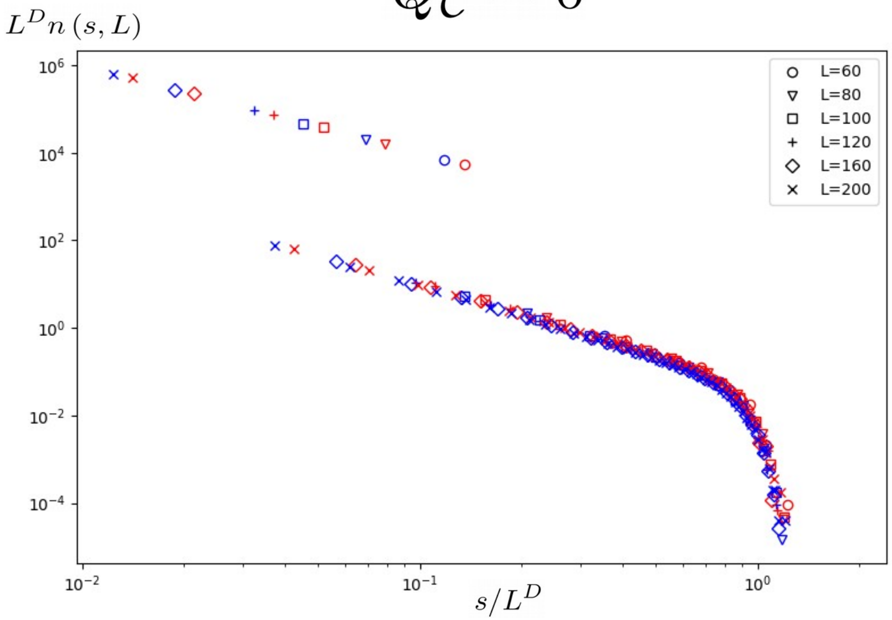
- Clusters seem to scale with fractal dimension $D = 1.88$
- Evidence of $1/x$ and $1/x^2$ scaling

Cluster size scaling of individual charges

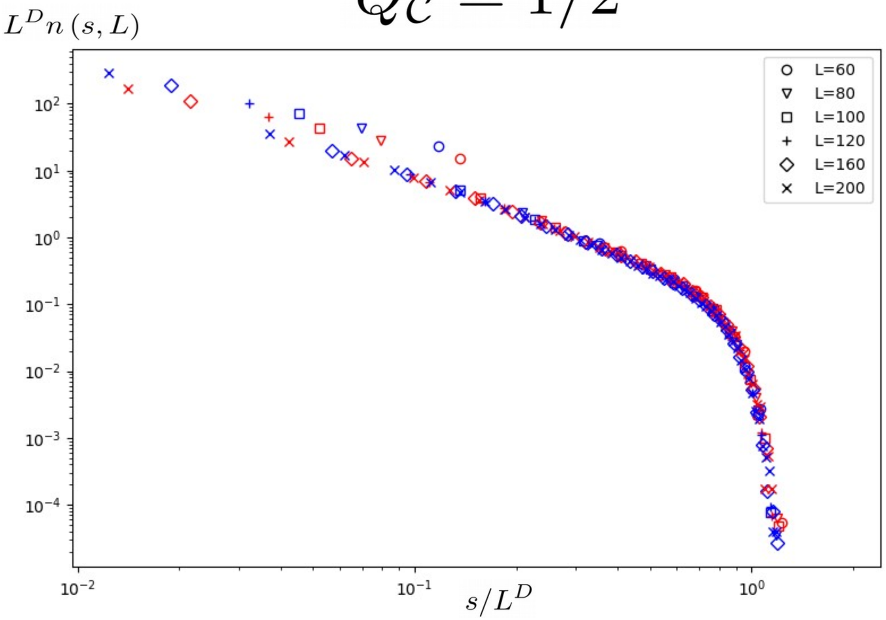
Neutral and meron clusters

Topological action
Constrain action

$Q_{\mathcal{C}} = 0$



$Q_{\mathcal{C}} = 1/2$

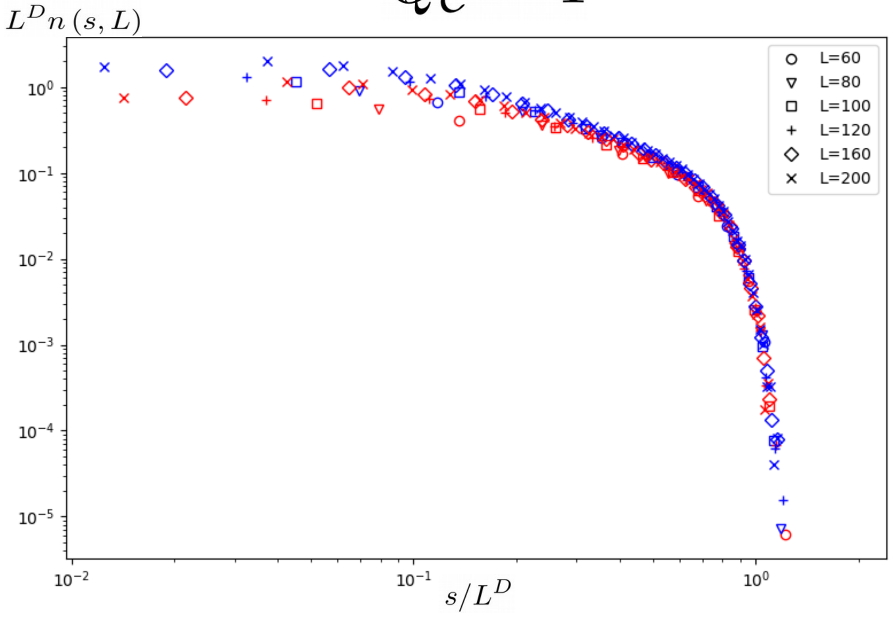


Cluster size scaling of individual charges

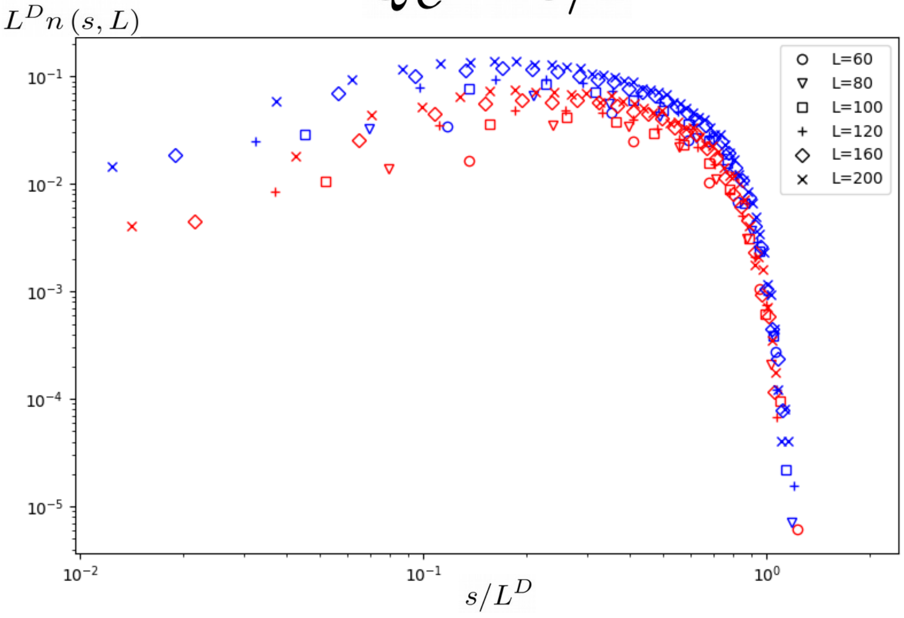
Higher charges

Topological action
Constrain action

$Q_{\mathcal{C}} = 1$



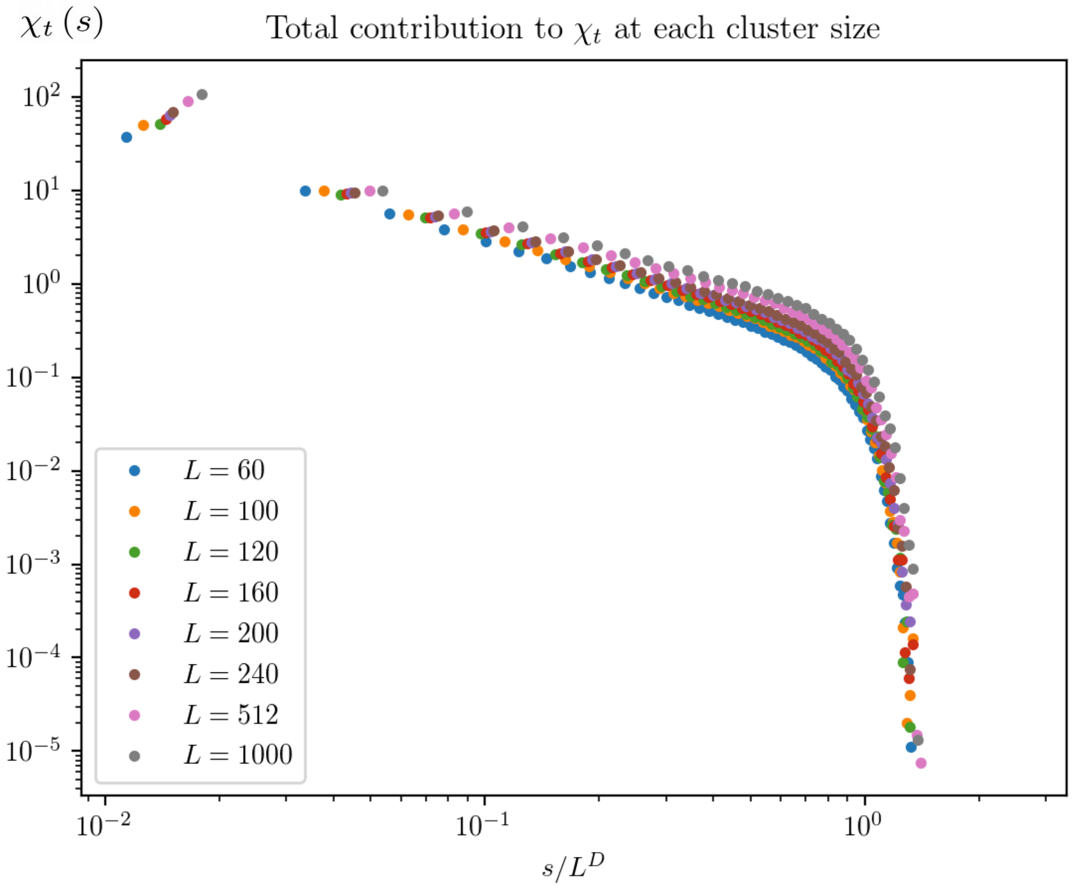
$Q_{\mathcal{C}} = 3/2$



...

Cluster size scaling universality

Contribution, at each size, for the topological susceptibility

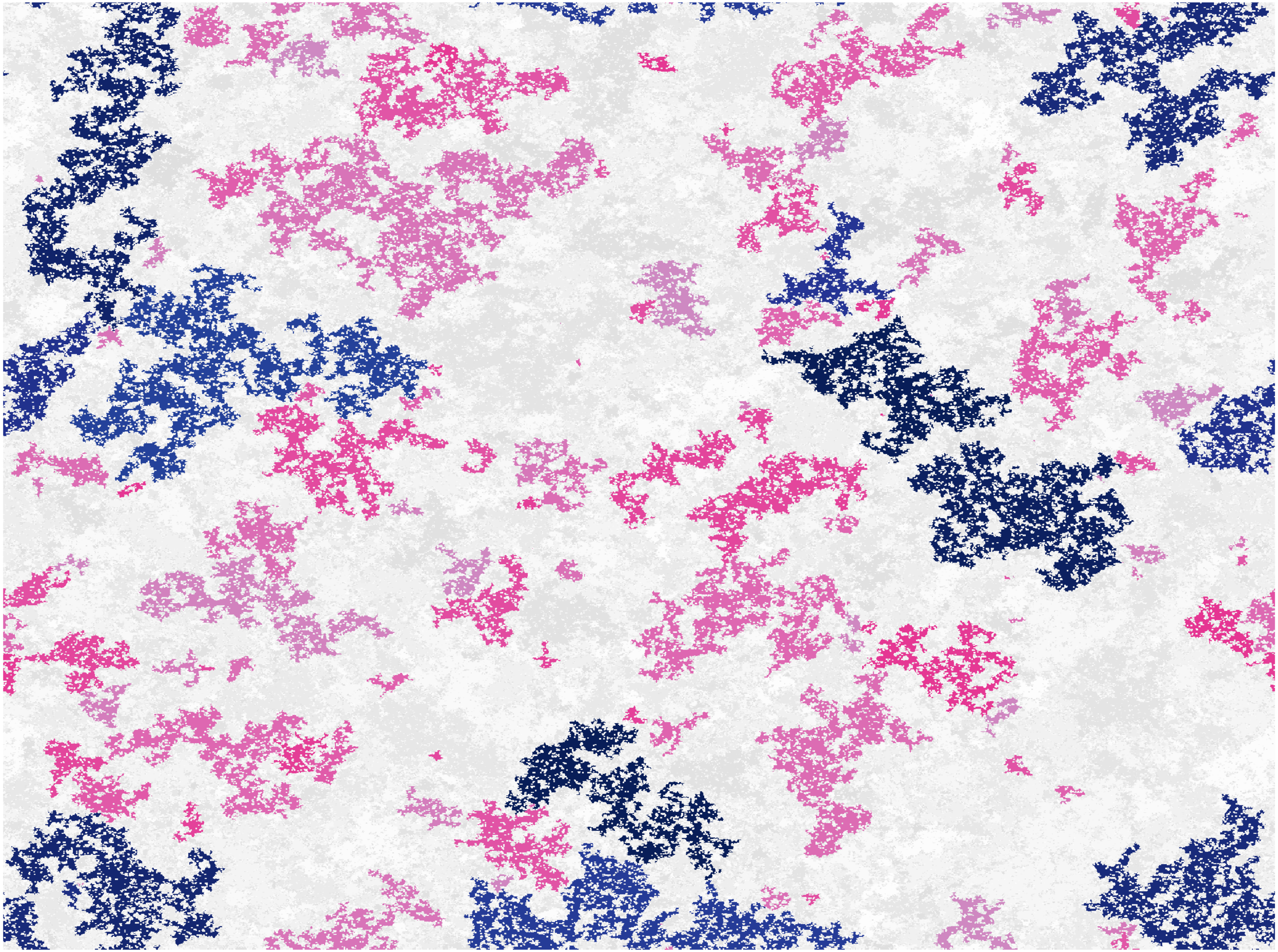


$$\chi_t = \frac{\langle Q^2 \rangle}{V} = \frac{1}{V} \sum_s \sum_{q \in \mathbb{Z}} q^2 n_q(s) \equiv \frac{1}{V} \sum_s \chi_t(s)$$

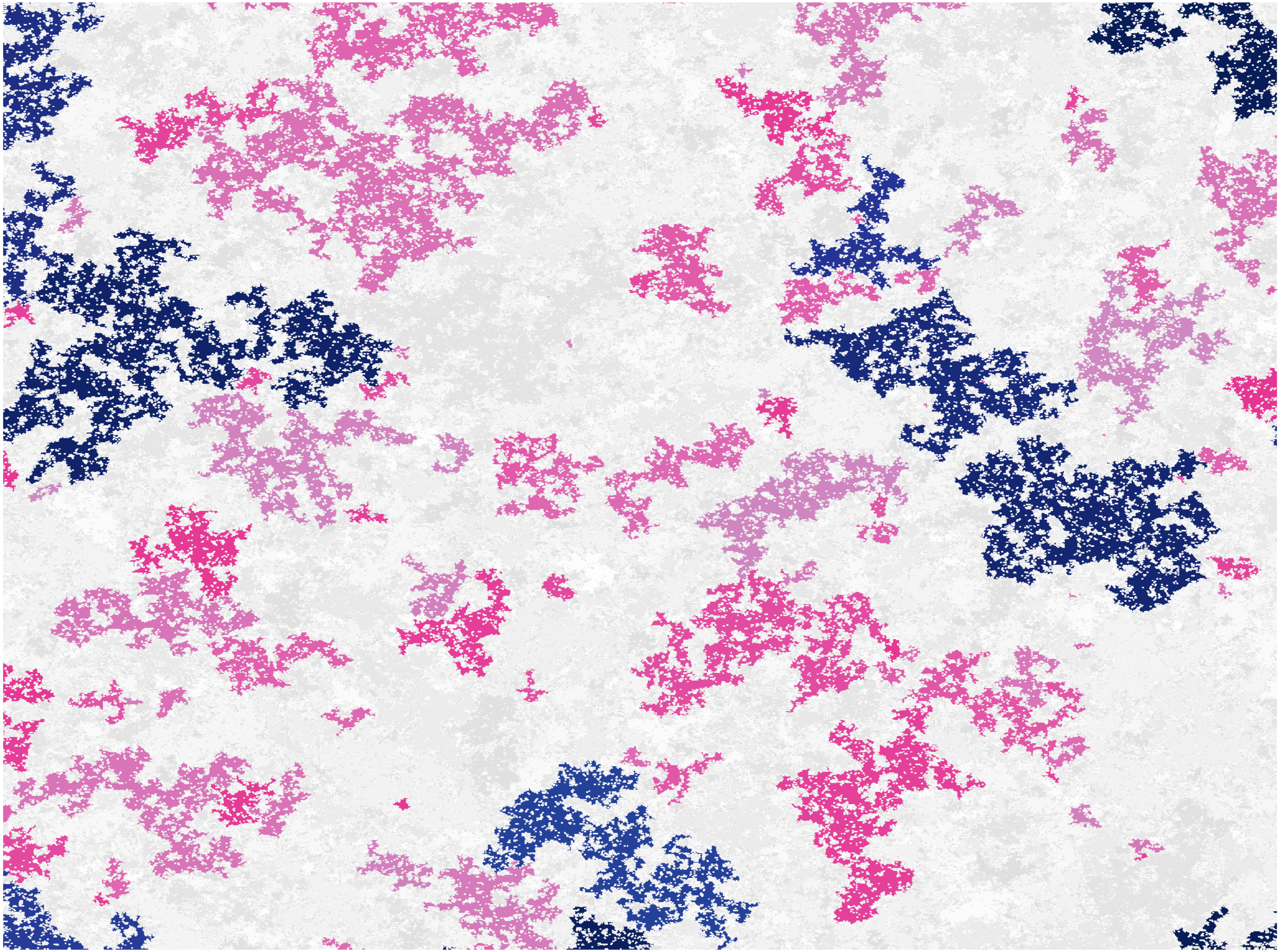
$$\boxed{\chi_t \sim (\log a)^2 + \log a}$$

- Up to finite size effects and lattice artifacts clusters scale like $1/x$
- There is a further logarithm drifting of the distributions.

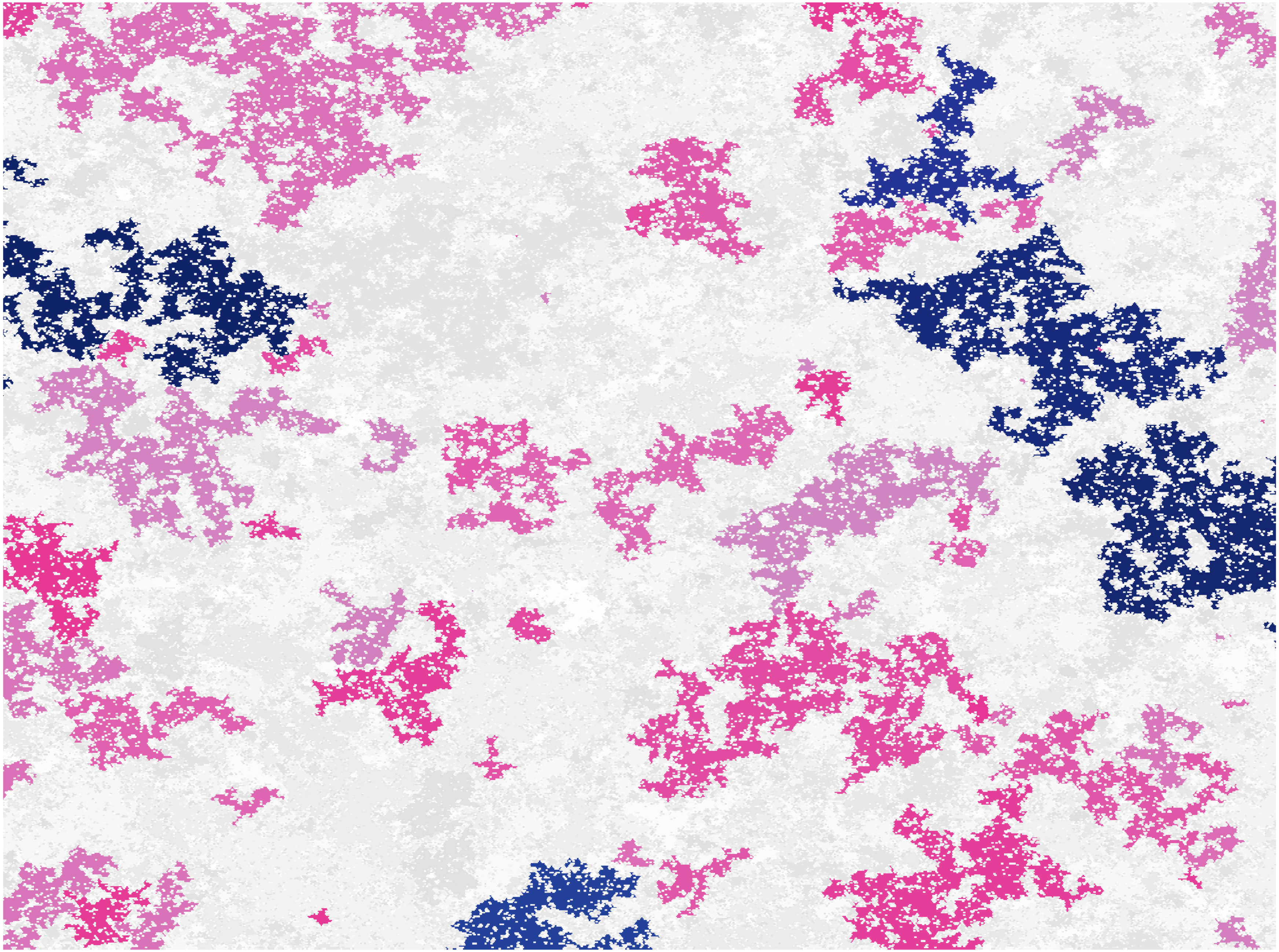
Clusters snapshot



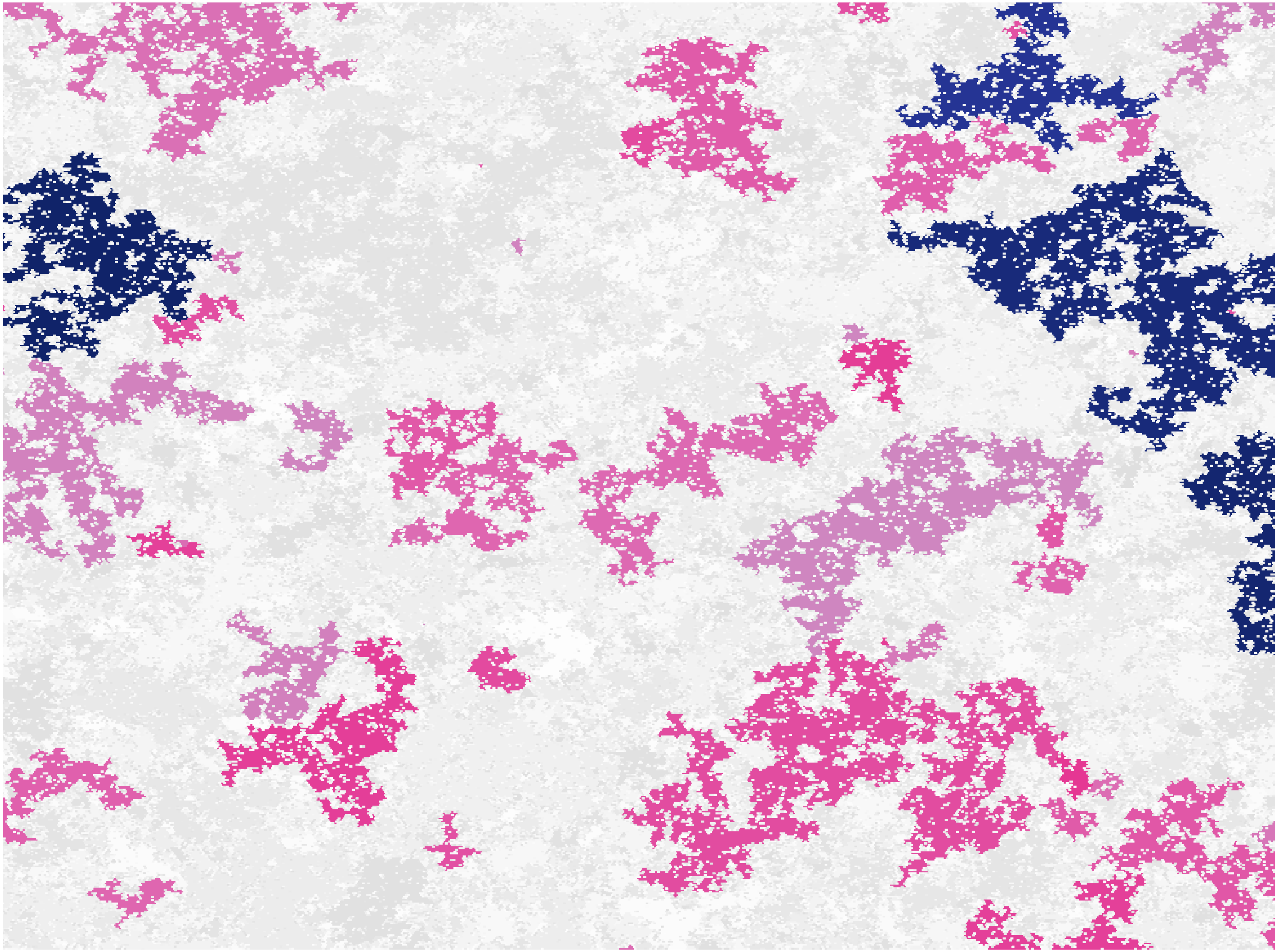
Clusters snapshot



Clusters snapshot



Clusters snapshot



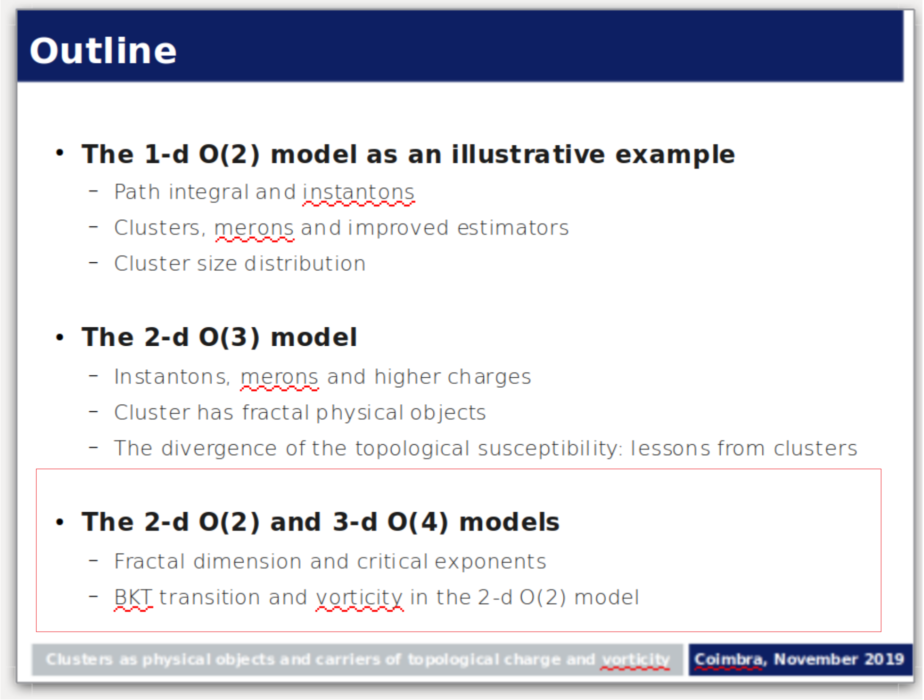
Clusters snapshot

An overview over the 2-d O(3) model

Cluster size scaling

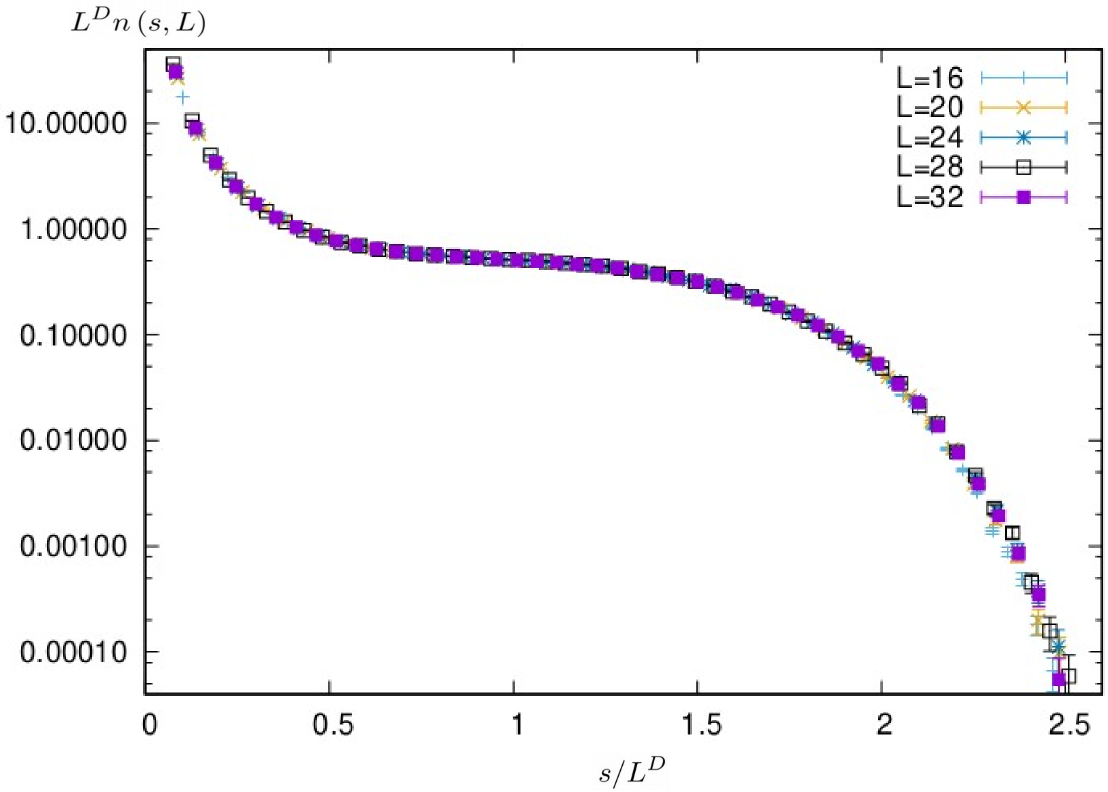
- If we do not distinguish clusters by their charge, the distribution still scales
- We find a log square divergence and identified its origin in terms of cluster distributions

What about other O(N) models?



The 3-d O(4) model

- Phase transition which shares universal features with the finite temperature chiral phase transition present in 2-flavor QCD in the chiral limit
- Non trivial topology



- The fractal dimension is given by

$$D = 2.485$$

In the Ising model one can prove analytically that the fractal dimension is related to critical exponents.

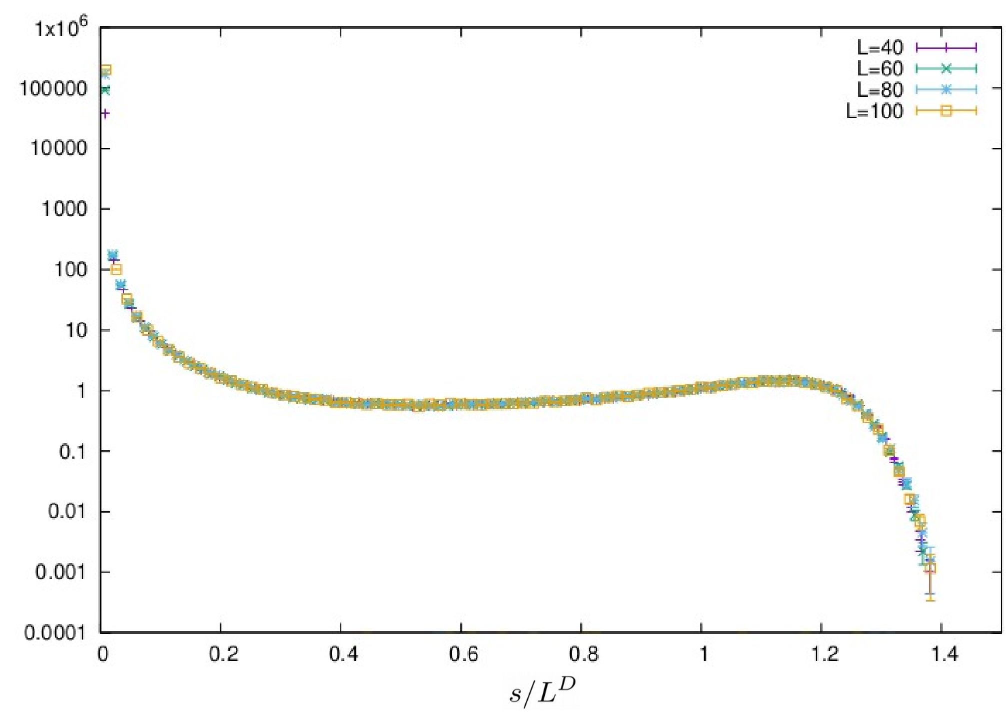
$$D = d - \frac{\beta}{\nu}$$

Scaling law still works!

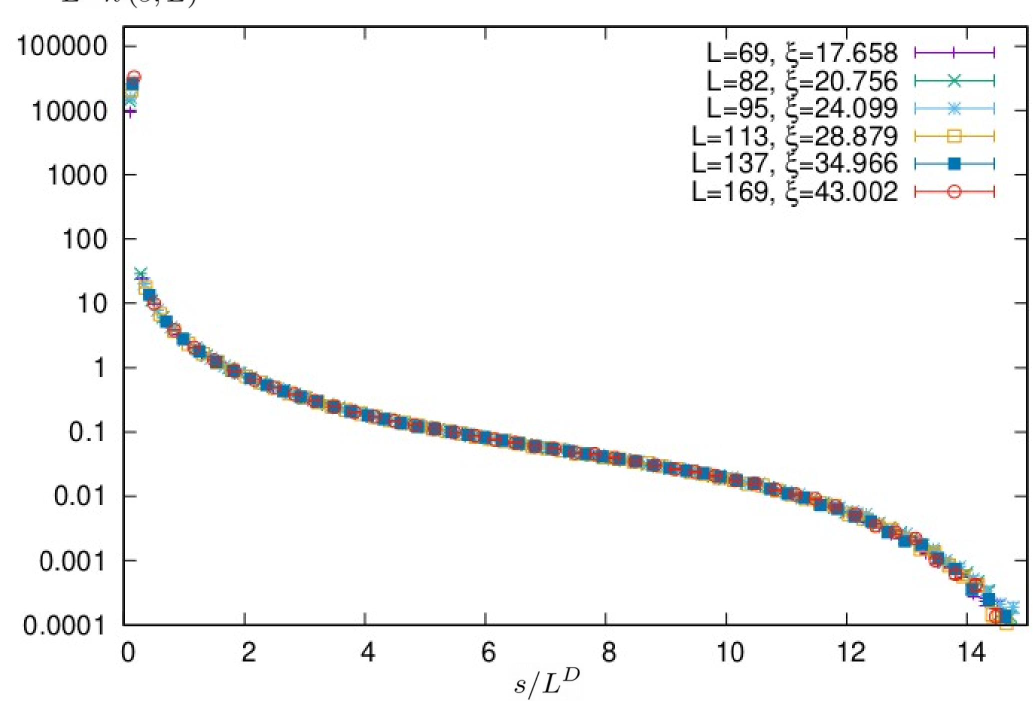
The 2-d O(2) model

- Exhibits a Berezinskii-Kosterlitz-Thouless (BKT) transition between a massless and a massive phase.
- In the massless phase, the exponent η varies continually with the coupling
- Has trivial topology but has vortices associated with plaquettes

Massless Phase



Massive Phase

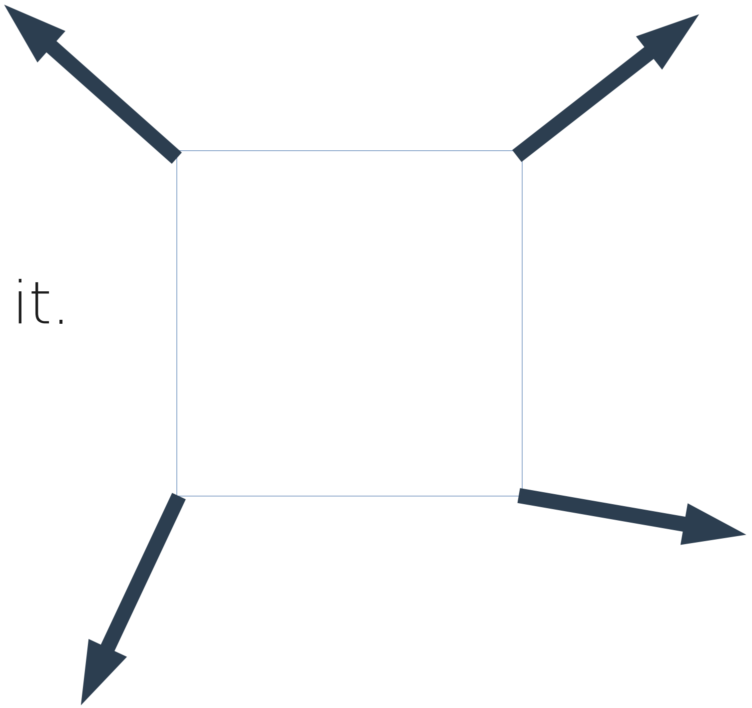


$$D = d - \frac{\beta}{\nu} \longrightarrow D = d - \frac{\eta}{2}$$

Scaling law still works, again!

Clusters and vorticity

A cluster can never have the four sites of a vortex.



When a vortex exists there are always two spins that, when flipped alone, destroy it.

We can distribute the vorticity, at most, by two clusters.

We can classify clusters by their vorticity

Overview and what is to come

- Clusters are physical objects that exist in the continuum limit
- They are carriers of topology on models with non-trivial topology
- They provide analytical and numerical insights
  - log square divergence was previously observed in the 2-d O(3) model
- Clusters are not semi-classical objects
- Fractal dimension of clusters associated with critical exponents
- Clusters can be carriers of vorticity

What lies ahead:

- Can we identify topological charge carriers in other theories?
- Can we explain, quantitatively, the BKT transition using clusters as carriers of vorticity?

THANK YOU!