

Scaling invariance breaking in Four-boson systems and beyond

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OUTLINE

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11. Summary

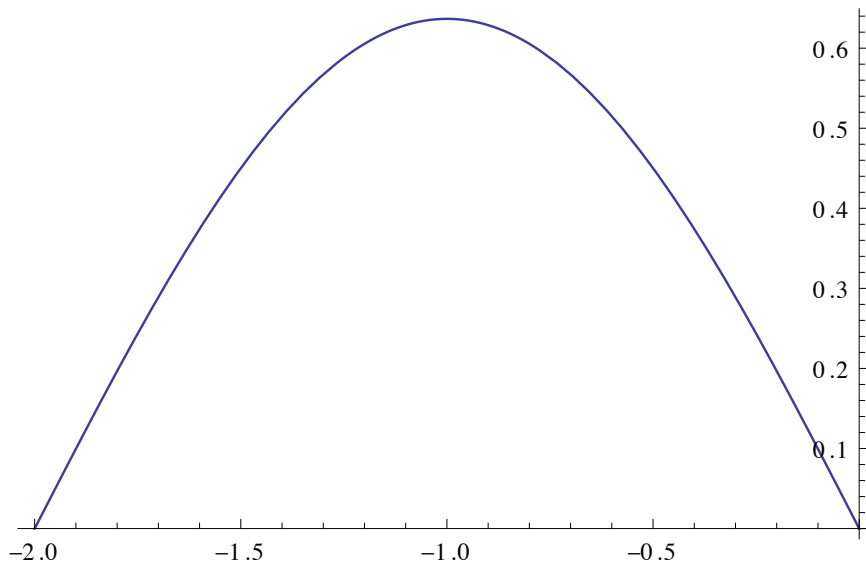
1. Scale invariance and Breaking: example integral equation

$$f(k) = \lambda \int_0^\infty dp p \frac{f(p)}{k^2 + p^2}$$

int. eq. invariant under: $k \rightarrow \xi k$ and $p \rightarrow \xi p$

$f(k)$ and $f(\xi k)$ solutions $f(k) = k^\eta$

transcendental eq.: $\lambda = -\frac{2}{\pi} \sin\left(\frac{\pi\eta}{2}\right)$



$$\lambda < \lambda_c = \frac{2}{\pi} \text{ for } -2 < \eta < 0$$

Maximum value

IF $\lambda > \lambda_c = \frac{2}{\pi}$ then $\eta = -1 + i s_0$

Solution: $f(k) = k^{-1} \sin(s_0 \log k/k^*)$

Continuous symmetry breaking to a discrete one & k^ dimensional scale*

$$k \rightarrow \exp(n\pi/s_0) k$$

Applies to 3bosons, 4bosons @unitarity, relativistic bound states (Bethe-Salpeter eq.)

Scale invariance and Breaking: example $1/r^2$ potential

$$-\frac{d^2}{dr^2}\psi(r) - \frac{K}{r^2}\psi(r) = E\psi(r)$$

Continuous symmetry breaking to a discrete one & r^* dimensional scale

$$\psi(r)|_{r\sqrt{E}\ll 1} \rightarrow r^{\frac{1}{2}} \sin(\sqrt{4K-1} \log(r/r^*))$$

$$K > \frac{1}{4} \text{ discrete scale symmetry}$$

(”fall to the center” Landau Quantum Mechanics)

Efimov effect & Thomas collapse

2. Three-boson system: Subtle three-body phenomenon in $L_{\text{total}}=0$:

The Efimov effect

Nuclear Physics

Vitaly Efimov Phys. Lett. B 33, 563 (1970).

Vol 440/16 March 2006/doi:10.1038/nature04626

nature

LETTERS

Evidence for Efimov quantum states in an ultracold gas of caesium atoms

T. Kraemer¹, M. Mark¹, P. Waldburger¹, J. G. Danzl¹, C. Chin^{1,2}, B. Engeser¹, A. D. Lange¹, K. Pilch¹, A. Jaakkola¹, H.-C. Nägerl¹ & R. Grimm^{1,3}

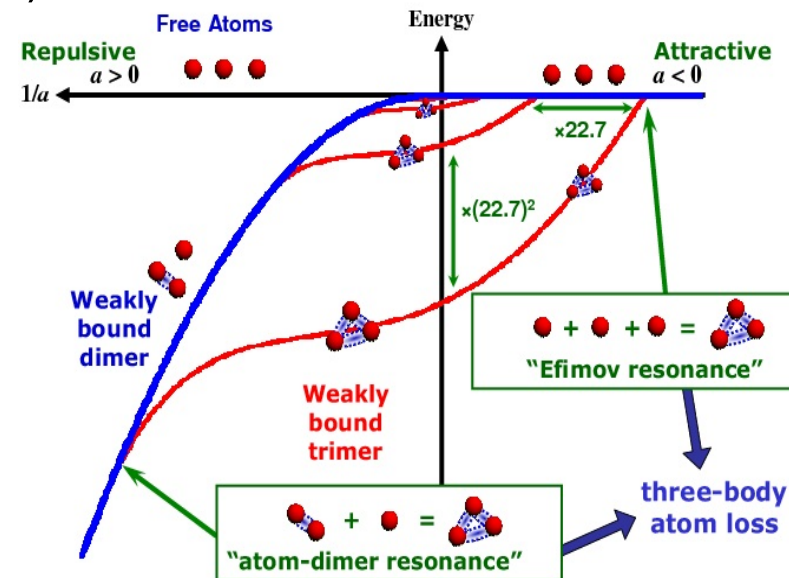
nature
physics

LETTERS

PUBLISHED ONLINE: 22 FEBRUARY 2009 | DOI: 10.1038/NPHYS1203

Observation of an Efimov-like trimer resonance in ultracold atom-dimer scattering

S. Knoop^{1*}, F. Ferlaino¹, M. Mark¹, M. Berninger¹, H. Schöbel¹, H.-C. Nägerl¹ and R. Grimm^{1,2}



2. Three-bosons: Thomas and Efimov effects

L. H. Thomas, Phys. Rev. 47, 903 (1935)

V. Efimov, Phys. Lett. B 33, 563 (1970).

Thomas collapse (1935)	Efimov effect (1970)
$r_o \rightarrow 0$	$ a \rightarrow \infty$
Route to collapse!	infinitely many bound states condensing at $E=0$

$$|a|/r_o \rightarrow \infty$$

Adhikari, Delfino,TF,Goldman,Tomio, PRA37 (1988) 3666

One three-body scale is necessary to represent short-range physics !!!!

Jensen, Riisager, Fedorov, Garrido, RMP76, 215 (2004)

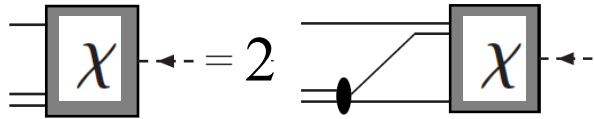
Braaten, Hammer Phys. Rep.428, 259 (2006)

Qualitative view Thomas-Efimov effect

- State of maximum symmetry in s-wave
- Effective interaction $V(\rho)$ hyper-radius $\rho = \left(\sum r_{ij}^2\right)^{\frac{1}{2}}$
$$-\frac{d^2}{d\rho^2}\psi(\rho) - V(\rho)\psi(\rho) = 0$$
- When $a^{-1} \rightarrow 0$ no dimensional scale!!!!
- $V(\rho) = -\frac{K}{\rho^2}$ if $K > 1/4$ $\psi(\rho) \sim \sin(s_0 \log \rho/\rho^*)$
- Efimov factor $s_0 = 1.00624$
- Three-boson energies $E_{n+1} = E_n e^{-2\pi/s_0}$ ***Discrete symmetry!***

3. Three-Bosons: SKM equations & Danilov's UV solution

Skorniakov and Ter-Martirosian equations (1956)



Dirac-delta interaction

$$\chi(\vec{k}) = \frac{\pi^{-2}}{\pm\sqrt{-E_2} - \sqrt{-E_3 + \frac{3}{4}k^2}} \int d^3p \left(\frac{1}{E_3 - k^2 - p^2 - \vec{p} \cdot \vec{k}} + \frac{1}{\mu_3^2 + k^2 + p^2 + \vec{p} \cdot \vec{k}} \right) \chi(\vec{p})$$

Adhikari, TF, Goldman, PRL74, 487 (1995); Adhikari, TF, *ibid.* 74, 4572 (1995)

$\hbar = m = 1$ (+) dimer bound & (−) dimer virtual

Thomas collapse: E_2 finite & $\mu_3 \rightarrow \infty$ & $\epsilon_2 = E_2/(\mu_3)^2 \rightarrow 0$

Efimov effect: $E_2 \rightarrow 0$ & μ_3 finite & $\epsilon_2 = E_2/(\mu_3)^2 \rightarrow 0$

Thomas-Efimov effect!

Scale invariance at the unitary limit and breaking

G.S. Danilov, Sov. Phys. JETP 13 (1961) 349

$\epsilon_2 = \epsilon_3 = 0$ and $\mu_3 \rightarrow \infty$ or scatt lengths $\rightarrow$ infinity

s-wave:
$$\chi(y) = \frac{4}{\pi\sqrt{3}y} \int_0^\infty dx x^2 \chi(x) \int_{-1}^1 dz \frac{1}{x^2 + y^2 + x y z}$$

Solution:
$$\chi(y) = y^{s-2}$$

Efimov equation:

$$1 = \frac{8}{\sqrt{3}s} \frac{\sin(\pi s/6)}{\cos(\pi s/2)} \quad s = \pm i s_0 \quad s_0 \approx 1.00624$$

$$\chi(y) = a_+ y^{i s_0 - 2} + a_- y^{-i s_0 - 2}$$

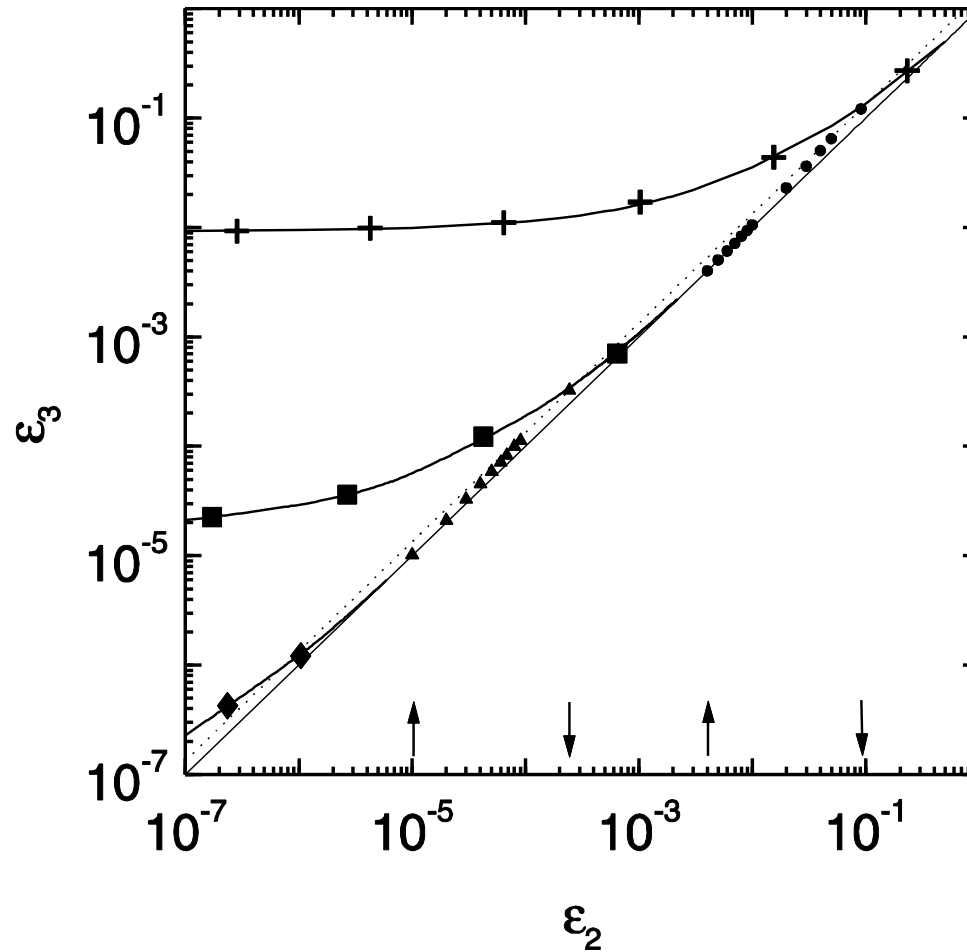
$$\chi(y) = y^{-2} \sin(s_0 \ln y + c)$$

One parameter to fix the solution $\rightarrow$ 3-body scale

Continuum scale invariance broken to a discrete one

Efimov effect

Efimov States – Bound and virtual states (3 identical bosons) (3D)



Bound-states (lines with simbols)

With Plus - Fundamental state

With squares – 1st. excited

With diamonds – 2nd. excited

Virtual-states (just simbols)

circles – 1st. Excited state

triangles – 2nd. Excited state

ε_2 bound

↓ the virtual state starts (dotted line)

↑ Virtual-state turns to an excited state (continuous line)

4. Limit cycles: scaling plots

$$\epsilon_3^{(N)} \equiv \epsilon_3^{(N)} (\pm \sqrt{\epsilon_2})$$

$$\xi \equiv \pm \sqrt{\epsilon_2} = \pm (E_2 \epsilon_3^{(N)} / E_3^{(N)})^{1/2}$$

$$\frac{E_3^{(N+1)}}{E_3^{(N)}} = \lim_{N \rightarrow \infty} \frac{\epsilon_3^{(N+1)}(\xi)}{\epsilon_3^{(N)}} = \mathcal{F} \left(\pm \sqrt{\frac{E_2}{E_3^{(N)}}} \right)$$

Scaling function

$$\mathcal{F}(0) = e^{2\pi/s_0} = 1/515$$

Efimov 1970

Scaling limit:

Frederico et al PRA60 (1999)R9

Yamashita et al PRA66(2003)052702

Limit cycle:

Bedaque, Hammer, van Kolck, PRL 82 (1999) 463

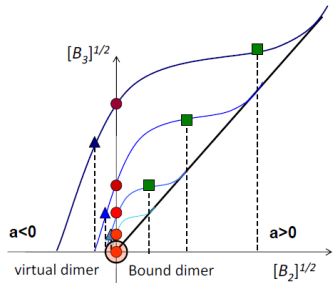
Mohr et al Ann. Phys. 321 (2006) 225

Scaling function & Limit Cycle

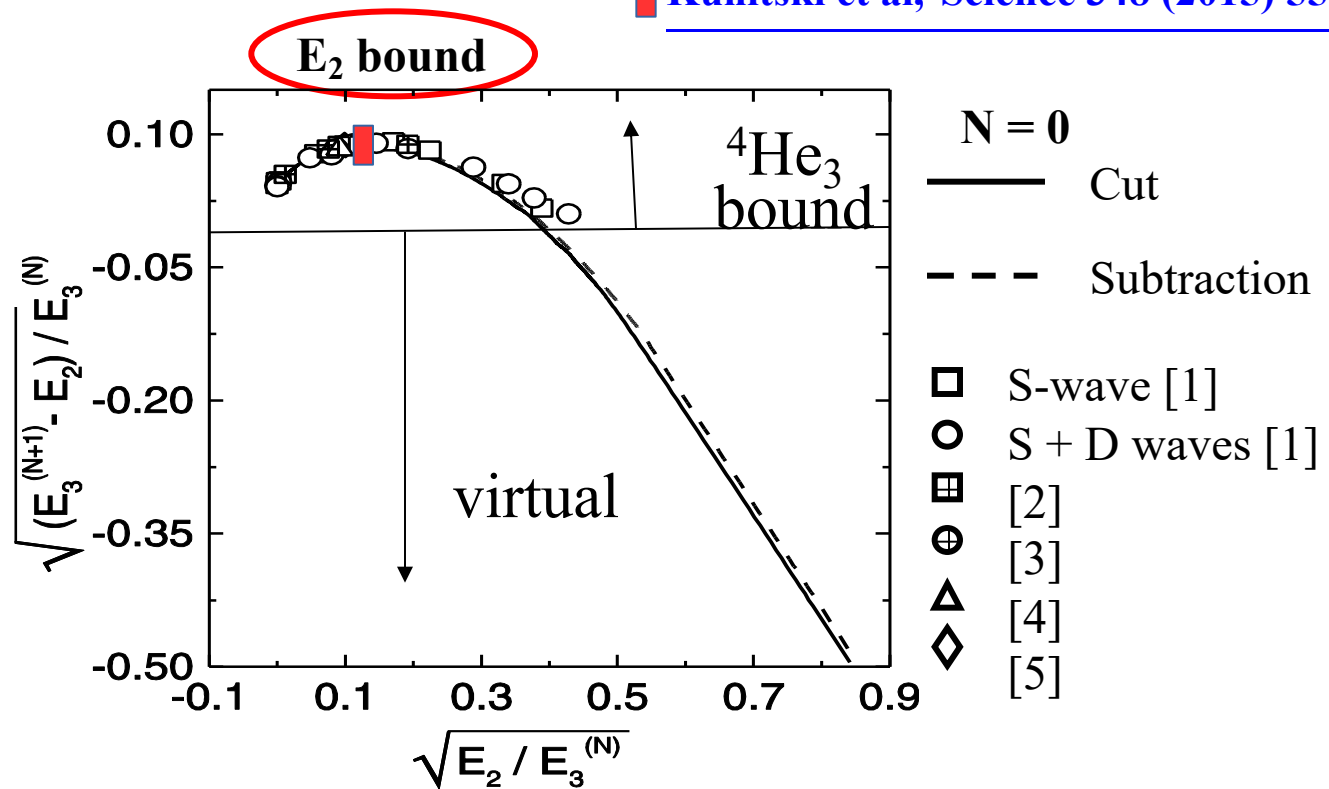
T. Frederico, LT, A. Delfino and E. A. Amorim. *Phys. Rev.* **A60**, R9 (1999).

Yamashita et al PRA66(2003)052702

Kunitski et al, Science 348 (2015) 551



Scaling plot



[1] Th. Cornelius e W. Glöckle. *J. Chem. Phys.* **85**, 1 (1996).

[2] S. Huber. *Phys. Rev.* **A31**, 3981 (1985).

[3] P. Barletta e A. Kievsky. *Phys. Rev.* **A64**, 042514 (2001).

[4] D. V. Fedorov e A. S. Jensen. *J. Phys.* **A34**, 6003 (2001).

[5] E. A. Kolganova, A. K. Motovilov e S. A. Sofianos. *Phys. Rev.* **A56**, R1686 (1997).

Range correction: Thogersen, Fedorov, Jensen PRA78(2008)020501(R)

5. Mass imbalanced systems

Generalization of Danilov's method to AAB systems @ unitary limit

M. T. YAMASHITA *et al.* PHYSICAL REVIEW A **87**, 062702 (2013)

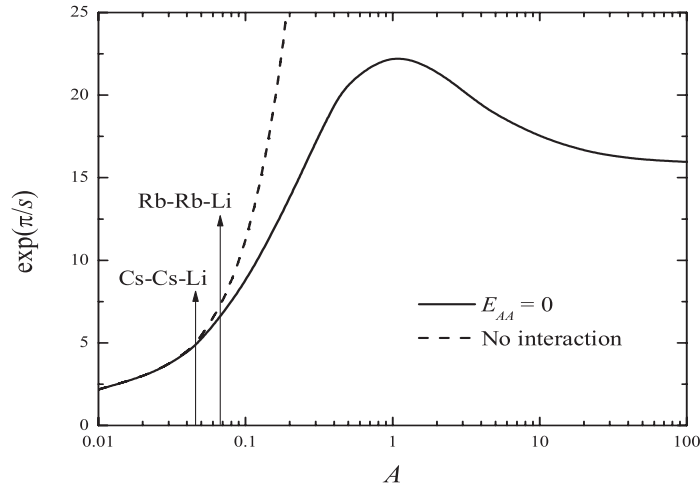


FIG. 1. Scaling parameter s as a function of $\mathcal{A} = m_B/m_A$ for $E_{AA} = 0$ and $E_{AB} = 0$ (resonant interactions) (solid line) and for the situation where $E_{AB} = 0$ but with no interaction between AA (dashed line). The arrows show the corresponding mass ratios for ^{133}Cs - ^{133}Cs - ^6Li and ^{87}Rb - ^{87}Rb - ^6Li .

A. S. Jensen and D. V. Fedorov, *Europhys. Lett.* **62**, 336 (2003).

Observed 3B Recombination peaks in atomic traps (Heidelberg group)

^{133}Cs - ^{133}Cs - ^6Li

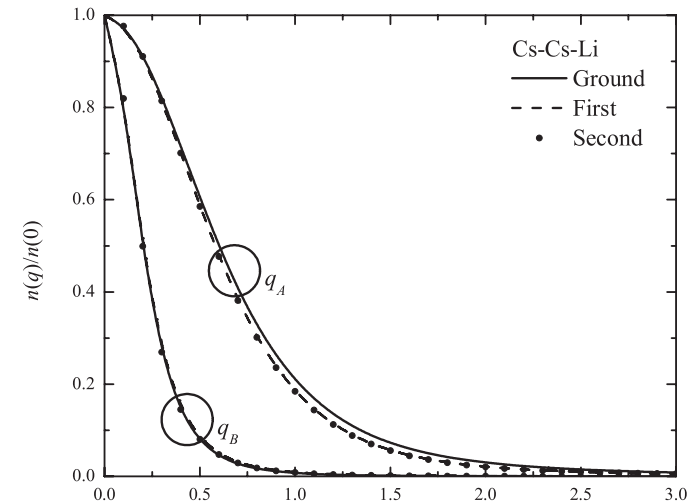
Pires et al. PRL112(2014)250404 (a < 0)

Ulmanis et al PRL117(2016)153201 (a > 0)

$$\frac{\mathcal{A}}{\pi} \left(\frac{\mathcal{A} + 1}{2\mathcal{A}} \right)^{3/2} \sqrt{\frac{2(\mathcal{A} + 1)}{\mathcal{A} + 2}} I_1(s) = 1$$

$$\mathcal{A} = m_B/m_A$$

M. T. YAMASHITA *et al.*



The momentum distributions for the particles A and B are

$$n(q_B) = \int d^3 p_B |\langle \vec{q}_B \vec{p}_B | \Psi \rangle|^2,$$

$$n(q_A) = \int d^3 p_A |\langle \vec{q}_A \vec{p}_A | \Psi \rangle|^2 \quad (10)$$

6. Dimensional reduction $D = 3 \rightarrow 2$

- ❖ *No Efimov effect in 2D! Bruch & Tjon PRA 19 (1979) 425*
- ❖ Compactification (periodic bound. conditions) $3D \rightarrow 2D \rightarrow 1D$ (3body)
Sandoval et al, JPB 51 (2018) 065004
- ❖ Danilov's equations in fractional dimensions (3body)
Rosa, TF, Krein, Yamashita, PRA97, 050701(R) (2018)
- ❖ EFT compactification & dim reg $4D \rightarrow 3D \rightarrow 2D$, $4D \rightarrow 2D$ (2body)
Beane & Jafry, JPB52(2019) 035001

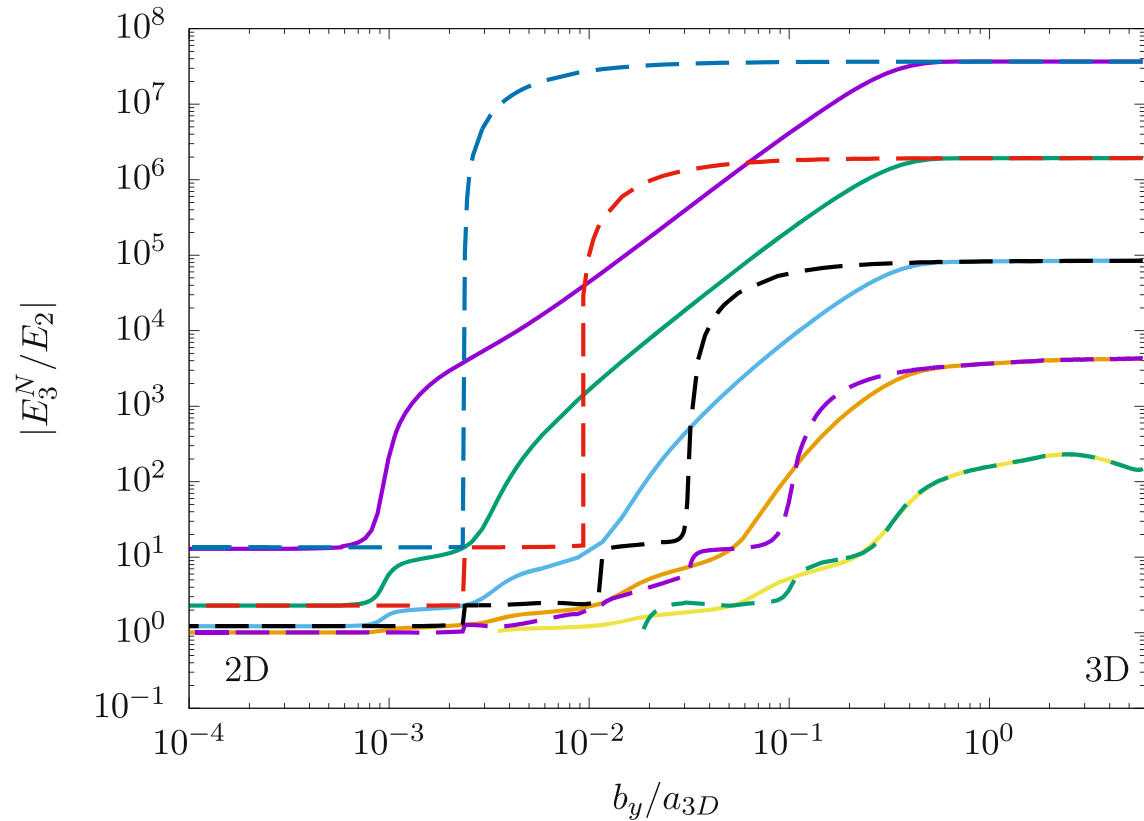


Figure 2. Trimer energies plotted in units of the two-body energy for $m_B/m_A = 6/133$ as functions of b_y/a_{3D} . For the solid lines the two-body energy varies with b_y while for the dashed lines it is kept constant (see text for discussion). Solid and dashed lines have different colors for visibility.

Danilov's equations in fractional dimensions AAB system

Rosa et al. PRA97, 050701(R) (2018)

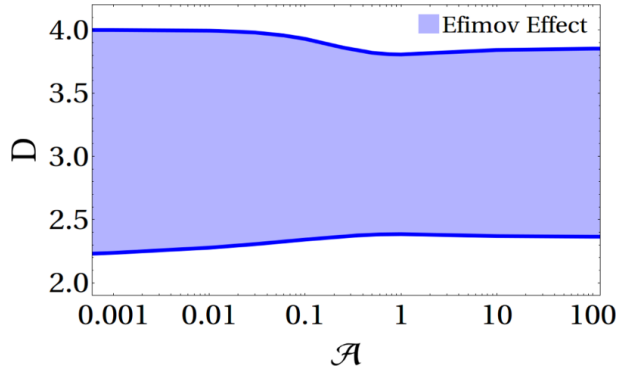


FIG. 1. Regions (in blue) where there is a real solution for the scaling factor s , solution to Eq. (8); outside this “dimensional band,” the Efimov effect does not exist. For $\mathcal{A} = 1$ we reproduce exactly the result in Ref. [7], where the dimensional limits are given by $2.3 < D < 3.8$.

$$\mathcal{A} = m_B/m_A$$

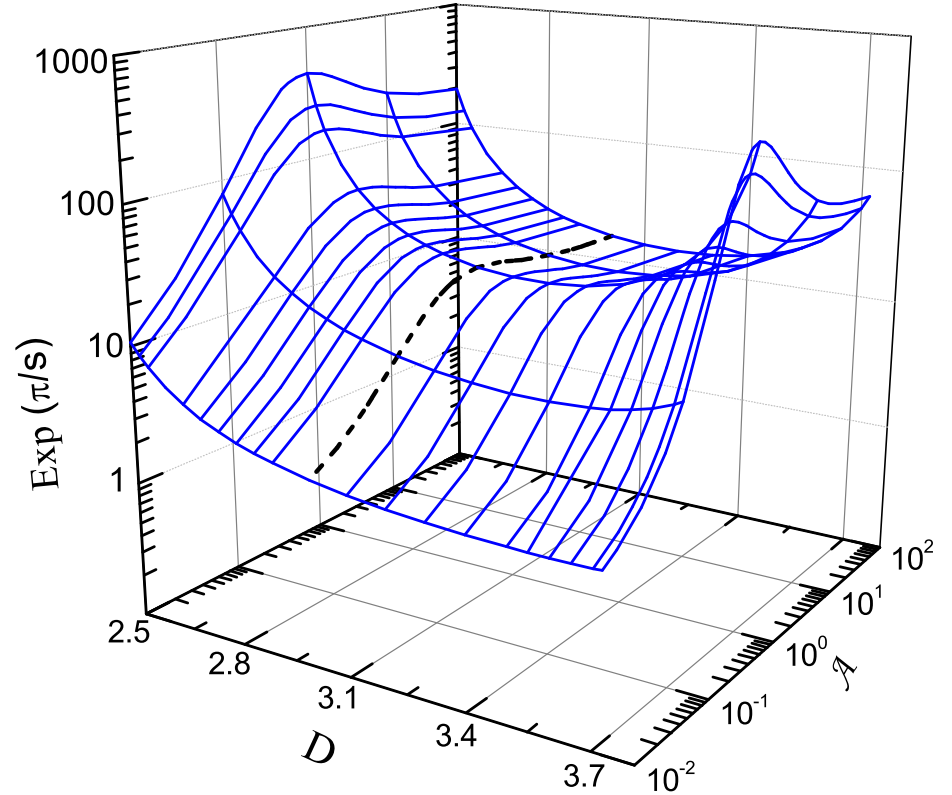
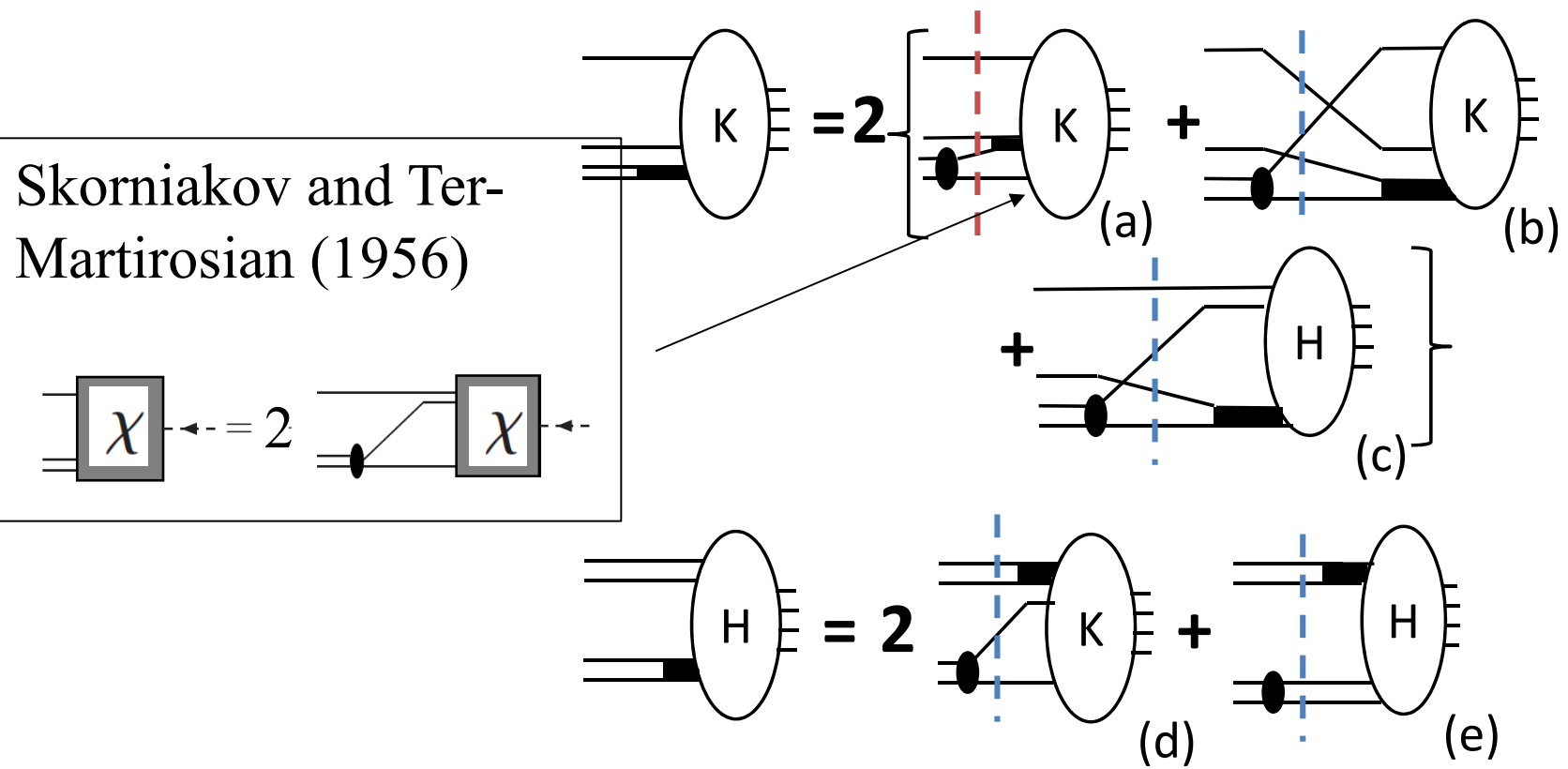


FIG. 2. Discrete scaling factor as a function of the mass ratio $\mathcal{A} = m_B/m_A$, and dimension D . The black dashed line shows the well-known situation of $D = 3$.

7. Faddeev-Yakubovski eqs (4-bosons), zero-range int., reg. and renor.: Scaling-plot

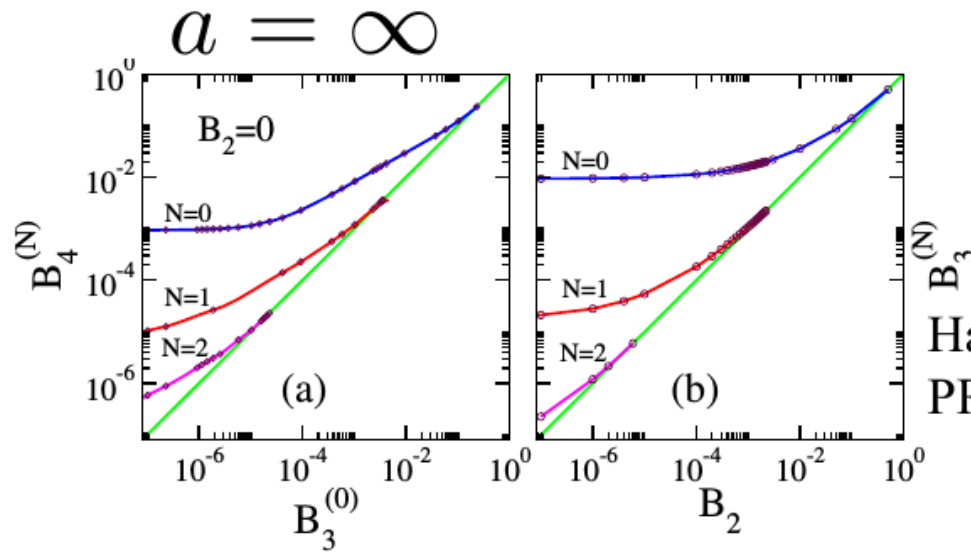
Collapse of the 4B system & 3B energy fixed

Yamashita, Tomio, Delfino, TF, EPL 75 (2006) 555



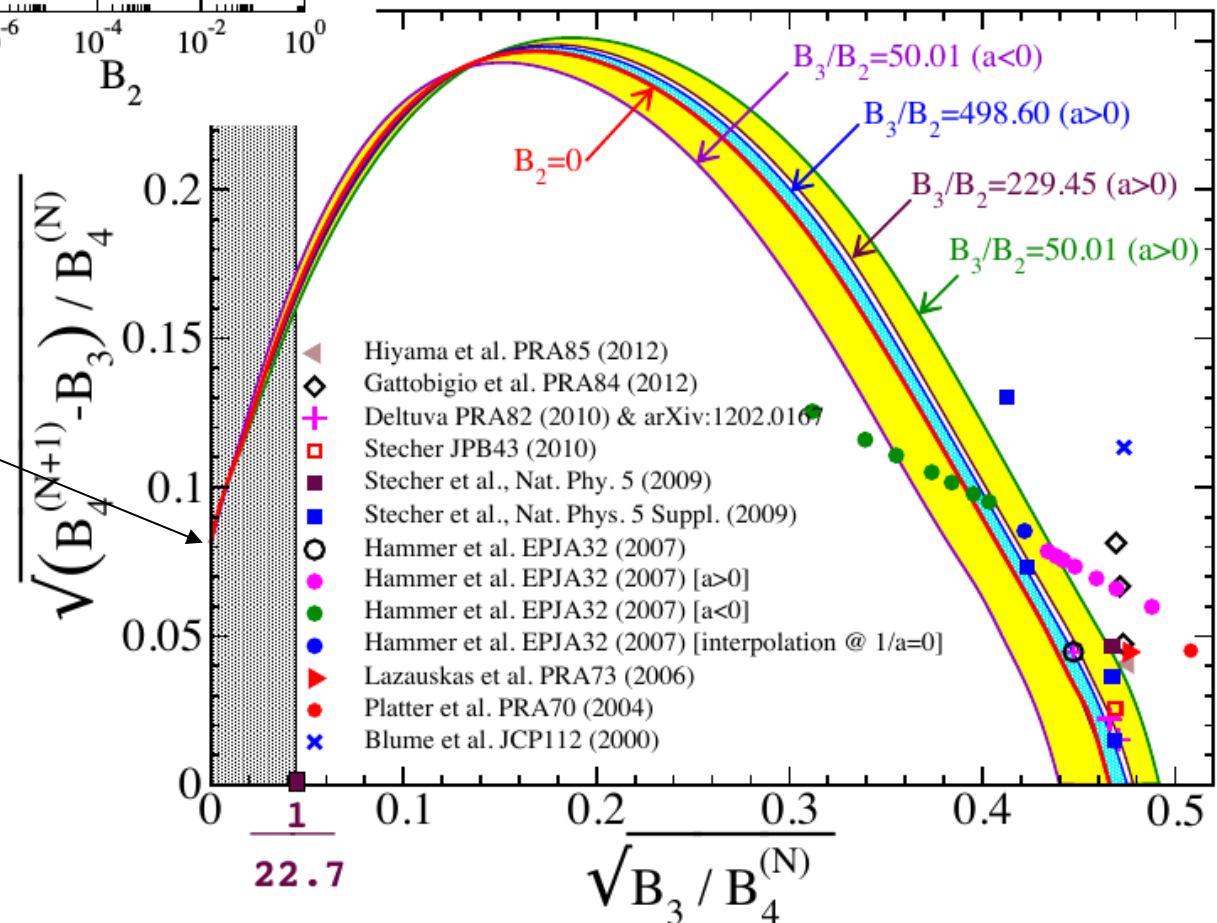
Subtracted Green's Functions: $G_0^{(N)} = \frac{1}{E-H_0} - \frac{1}{-\mu_N^2-H_0}$
with μ_3 (RED): 3B scale & μ_4 (BLUE): 4B scale

Scaling plot



Hadizadeh, Yamashita, Tomio, Delfino, TF,
PRL107, 135304 (2011)

$$B_4^{(N)} / B_4^{(N+1)} \sim 151$$



8. LLHH, LLLHH ... systems: B.O. approximation

LLHH Naidon, Few-Body Syst. 59, 64 (2018)

$$m_H \gg m_L \quad \textbf{\textit{L-H interaction only}}$$

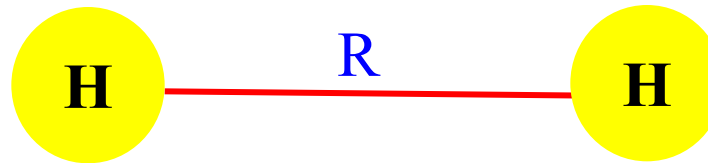
“Interwoven limit cycles in the spectra of mass imbalanced many-boson system“

De Paula, Delfino, TF, Tomio arXiv:1903.10321v1 [quant-ph]

Born-Oppenheimer approx.

Fonseca, Redish, Shanley, Nucl. Phys. A320 (1979) 273

Bhaduri, Chatterjee, van Zyl, Am. J. Phys. 79 (2011) 274-281; Am. J. Phys. 80 (2012) 94.



$$\left[\frac{d^2}{dR^2} + \frac{s_N^2 + \frac{1}{4}}{R^2} - \mathcal{B}_N \right] u = 0 \quad (N \geq 3)$$

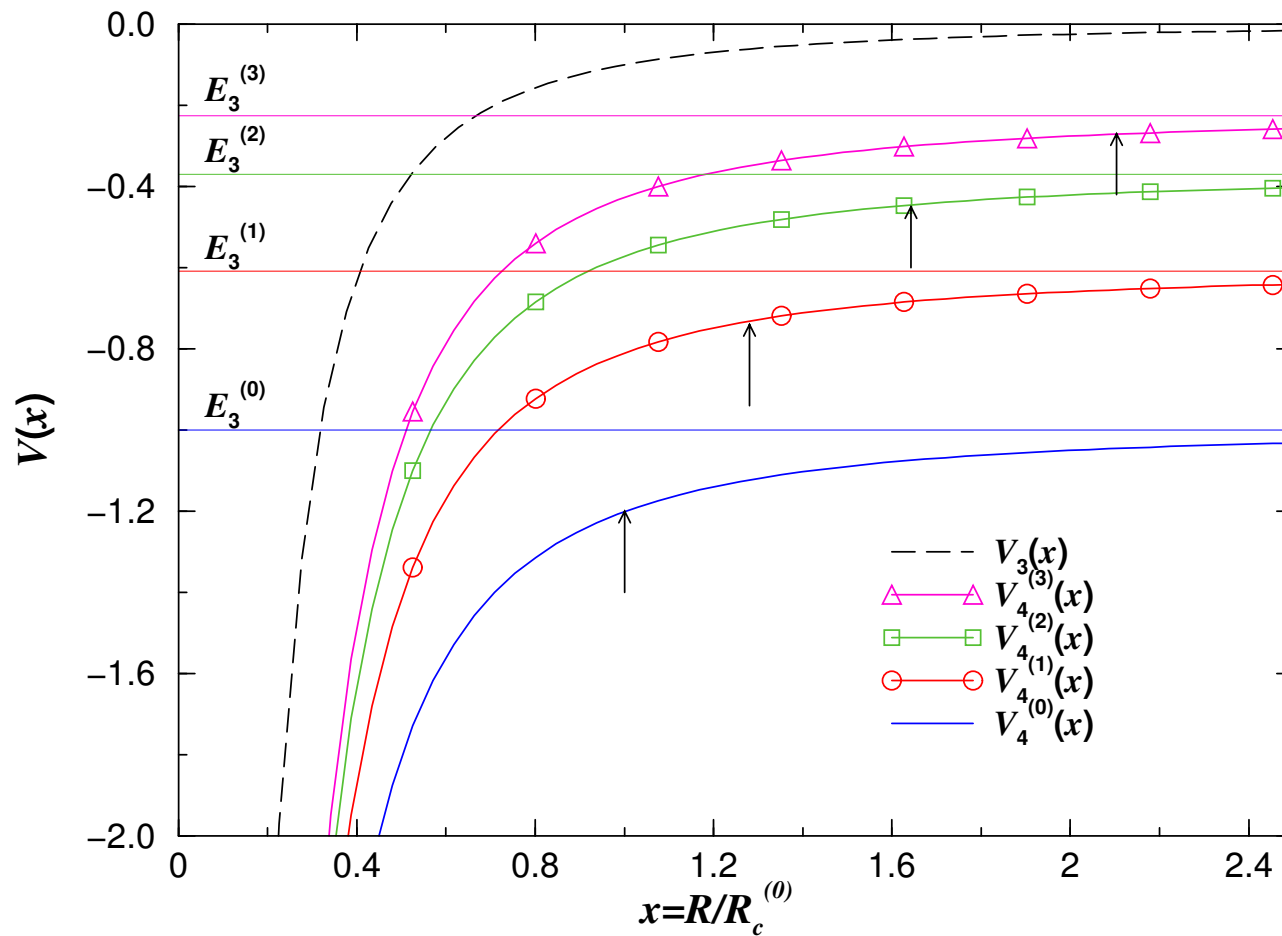
$$s_N \equiv s_N(A) \equiv \sqrt{\left(\frac{2+A}{4A}\right) (N-2)\gamma^2 - \frac{1}{4}} \quad A = m_L / m_H$$

$$\gamma = e^{-\gamma} = 0.5671433$$

$(N-2)$ Light bosons

New limit cycles beyond 3-body!

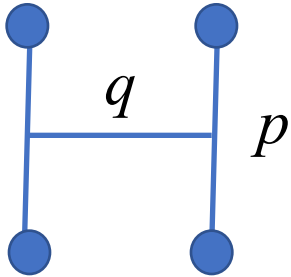
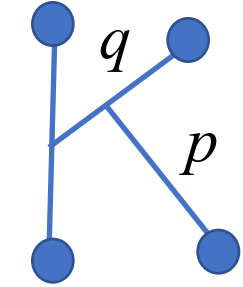
H-H B.O. potential 4-body system HHLL @ unitarity



(up-arrow) size of the three-body LHH cut the B.O. HH potential

9. FY scale invariance in UV & scale-invariance breaking: 4-body scale

“Four-Boson continuous scale symmetry breaking” TF, de Paula, Delfino, Tomio, FBS 2019



$$\begin{aligned} \mathcal{K}(q, p) = & \frac{2}{\sqrt{(3/4)q^2 + (2/3)p^2}} \int d\Omega dk k^2 \left\{ \frac{\mathcal{K}(k, p)}{q^2 + k^2 + (2/3)p^2 + q k z_0} \right. \\ & + \frac{27 \mathcal{K}(k, 3\sqrt{k^2 + p^2 - 2pkz_1})}{q^2 + (70/9)p^2 + 9k^2 - 16pkz_1 - (8/3)pqz_2 + 3kqz_0} \\ & \left. + \frac{\mathcal{H}(k, \sqrt{k^2/4 + p^2 - pkz_1})}{q^2 + (10/9)p^2 + k^2 + qkz_0 - (4/3)pkz_1 - (2/3)pqz_2} \right\}, \quad (2) \end{aligned}$$

$$\begin{aligned} \mathcal{H}(q, p) = & \frac{1}{\sqrt{q^2/2 + p^2}} \int d\Omega dk k^2 \left\{ \frac{\mathcal{H}(q, k)}{(q^2/2 + p^2 + k^2)} \right. \\ & \left. + \left(\frac{27}{4} \right) \frac{\mathcal{K}(k, (3/2)\sqrt{k^2 + q^2 + kqz_0})}{(3/2)q^2 + p^2 + (9/4)k^2 + 3qkz_0} \right\}, \quad (3) \end{aligned}$$

SCALE INVARIANT!

where k is within the interval $[0, \infty[$, $z_0 \equiv z_1 z_2 + \sqrt{(1 - z_1^2)(1 - z_2^2)} \cos \phi$, $d\Omega \equiv (2\pi)^{-2} dz_1 dz_2 d\phi$, with $-1 \leq z_{i=1,2} \leq 1$, and $0 \leq \phi \leq 2\pi$.

$$\mathcal{K}(tq, tp) = t^a \mathcal{K}(q, p) \text{ and } \mathcal{H}(tq, tp) = t^a \mathcal{H}(q, p)$$

Solution: UV FY eqs.

$$\mathcal{K}(q, p) = \frac{q^\eta}{\sqrt{\frac{3}{4}q^2 + \frac{2}{3}p^2}} f_K(q/p) \quad \text{and} \quad \mathcal{H}(q, p) = \frac{q^\eta}{\sqrt{\frac{1}{2}q^2 + p^2}} g_H(p/q)$$

Approximate UV form of FY eqs.:

3-boson eq. SKTM

$$f_K(x) = \frac{4x^{-\eta}}{\sqrt{3}} \int d\Omega dy \frac{y^{1+\eta}}{x^2 + y^2 + xy z_0} \left\{ f_K(y) + \left(\frac{1}{3}\right)^\eta f_K\left(\frac{1}{3}\right) + g_H\left(\frac{1}{2}\right) \right\}$$

$$g_H(x) = \frac{2}{\pi} \int_0^\infty \frac{dy y}{y^2 + x^2} \left\{ g_H(y) + 2 \left(\frac{2y}{3}\right)^\eta f_K\left(\frac{2}{3}\right) \right\}.$$

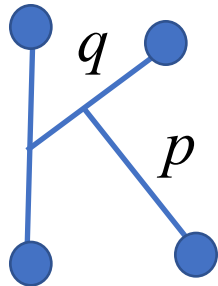
$f_K(x) = f_0 \quad \& \quad g_H(x) = x^\eta$

$$1 = \frac{8}{\sqrt{3}(\eta+1)} \frac{\sin\left[(\eta+1)\frac{\pi}{6}\right]}{\cos\left[(\eta+1)\frac{\pi}{2}\right]} \left\{ 1 - \frac{1}{3^\eta} \left[\frac{1 - \sin\left(\eta\frac{\pi}{2}\right)}{1 + \sin\left(\eta\frac{\pi}{2}\right)} \right] \right\}$$

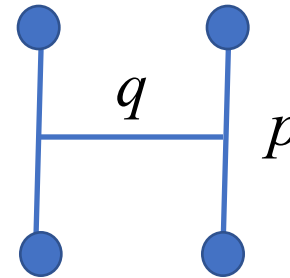
Scale symmetry breaking to a Discrete scale symmetry

Solution of the transcendental equation $\eta = -0.346 \pm 1.552i$

$$\mathcal{K}(q, p) \approx f_0 \frac{\sin(s_4 \log q / \Lambda^*)}{\sqrt{\frac{3}{4}q^2 + \frac{2}{3}p^2}} \quad \& \quad \mathcal{H}(q, p) \approx \frac{\sin(s_4 \log p / \Lambda^*)}{\sqrt{\frac{1}{2}q^2 + p^2}}$$



4-body scale



Compare to imaginary part: $1.25 i$ from Hadizadeh et al. PRL107 (2011) 135304

$$B_4^{(N)} / B_4^{(N+1)} = 151 = \exp(2\pi / s_4) \quad \& \quad s_4 = 1.25$$

In EFT 4-body scale @ NLO - Bazak et al. PRL 122 (2019) 143001

10. Scale inv. breaking in relat. bound states with Bethe-Salpeter eq.

Broad thoughts on scale symmetry breaking in QFT

- *Beyond non-relativistic physics*
- *Spontaneous Chiral symmetry breaking & Miransky scaling, N.Cim. A90(1985)149*
Kaplan, Lee, Son PRD80 (2009)12005
- *Relativistic bound states within Bethe-Salpeter approach:*

Fermions coupled to scalar, vector etc fields: coupling constant is dimensionless (QCD) - instabilities above critical value associated with log-periodic solutions... Efimov physics!

Fermion-fermion: Dorkin, Beyer, Semikh, Kaptari, Few Body Syst. 42 (2008) 1

Fermion-fermion: Carbonell, Karmanov, EPJA 46 (2010) 387

Fermion-boson: Alvarenga Nogueira, Gherardi, TF, Salmè, Colasante, Pace
PRD100 (2019)016021

Example: Fermion-boson Bethe-Salpeter equation $\frac{1}{2}^+$

Alvarenga Nogueira et al. in preparation

$$\Phi^\pi(k, p, J_z) = \left[O_1(k) \phi_1(k, p) + O_2(k) \phi_2(k, p) \right] U(p, J_z)$$

$$O_1(k) = \mathbb{I} \quad , \quad O_2(k) = \frac{\not{k}}{M} \quad , \quad (\not{p} - M) U(p, J_z) = 0$$

Ladder BSE in Euclidean space – vector exchange

$$k_4 = K \cos \varphi \quad \text{and} \quad k = K \sin \varphi \quad 0 < \varphi < \pi$$

$$-5 < \text{Real}[\eta] < -4$$

Maximum value of
the couplings product

$$\alpha_c = 1.18691\dots \quad \alpha^V = \frac{\lambda_F^v \lambda_S^v}{8\pi}$$

$$\phi_1(k_4, k) = K^{\eta+1} \quad \text{and} \quad \phi_2(k_4, k) = 0 \quad \eta = -4.08918\dots$$

$$\phi_1(k_4, k) = 0 \quad \text{and} \quad \phi_2(k_4, k) = K^\eta \quad \eta = -4.91082\dots$$

$$\psi_i(\xi, \gamma; \kappa^2) = iM \int_{-\infty}^{\infty} \frac{dk^-}{2\pi} \phi_i(k, p) \sim \gamma^{1+\frac{\eta_i}{2}}$$

Solution of the Ladder BS equation in Minkowski space via Nakanishi integral representation
[PRD100 (2019)016021]

$\mu/\bar{m}= 0.00$ - $B/\bar{m}= 0.500$ - $m_F= 1.00$ - $m_S= 1.00$ - $\xi_0= 0.50$

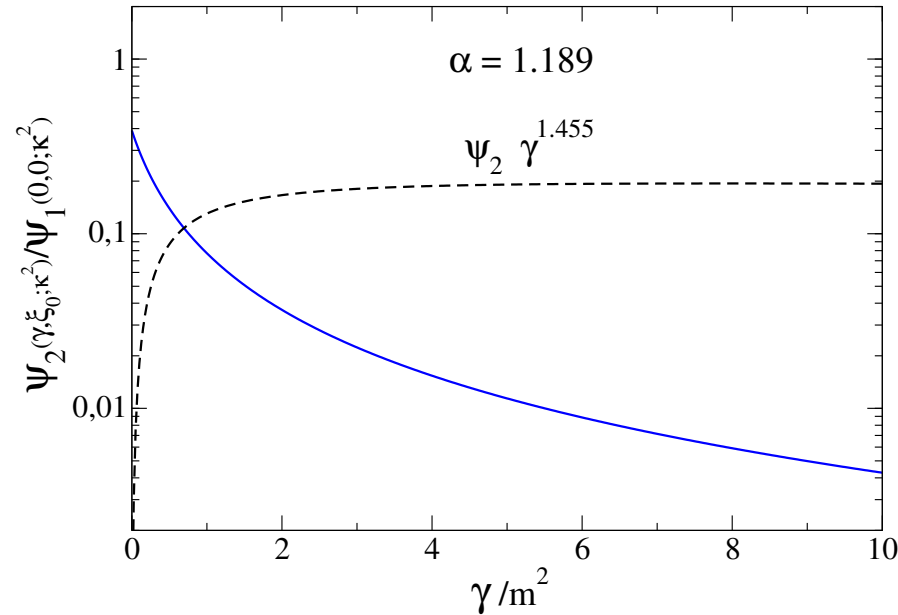


Figure 5.4. The light-front wave function $\psi_2(\gamma, z_0 = 0)$ obtained from the solution of the original equation (5.8) as a function of γ (solid blue curve) and its product with the asymptotic limit found in the high momentum limit (dashed black curve).

Summary

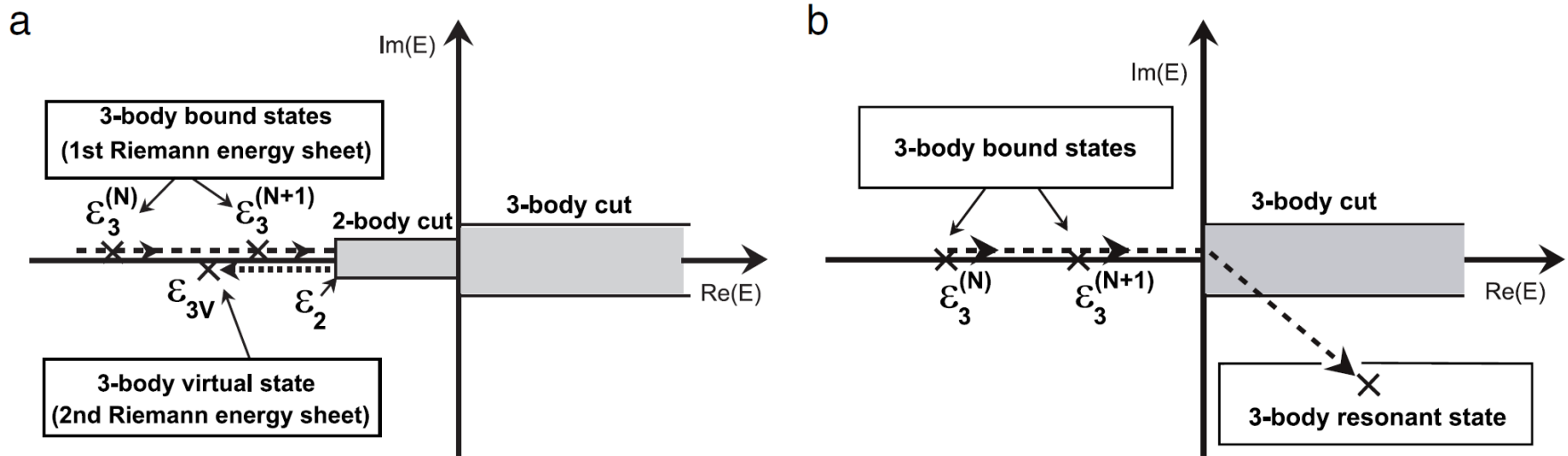
- ✓ *Scale invariance breaking & Efimov and Thomas effects;*
- ✓ *Dimensional reduction & suppression Thomas-Efimov effects;*
- ✓ *Scale invariance breaking in 4-boson systems: 4body scale;*
- ✓ *More particles with short-range interactions: B.O. approx. suggests new scales;*
- ✓ *Scale invariance breaking & Relativistic bound states.*

THANK YOU!

Thanks to:



Analytic structure & Efimov state trajectory



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Continuum resonances of Borromean systems: observation in atomic traps!

Resonant 3-body recombination (Innsbruck, Rice, Heidelberg, Bar Ilan, Florence...)