STABILITY AND CAUSALITY OF THE RELATIVISTIC THIRD ORDER HYDRODYNAMICS

Jobin Sebastian and Amaresh Jaiswal

School of Physical Sciences, National Institute of Science Education and Research Bhubaneswar, HBNI, India

Introduction

The hot and dense QCD medium created in relativistic heavy ion collisions behaves like a fluid system and successfully studied by tools of relativistic hydrodynamics. A theory of relativistic hydrodynamics should be stable and causal so that the disturbance in the fluid medium propagates with finite velocities. Causality is the restriction imposed by special theory of relativity which doesn't allow any information to travel faster than the speed of light. We present the analysis of the stability and causality property of the third-order relativistic viscous hydrodynamics developed in the Ref. [1].

Relativistic hydrodynamics

- Fluid mechanics is one of the oldest and most successful theories in the history of physics.
- We have an intuitive understanding about the theory in its non-relativistic form from the everyday experience with hydrodynamics of water.
- We define energy-momentum tensor for an ideal fluid, $T^{\mu\nu}$, which is a second rank symmetric tensor tells us all we need to know about the system, like energy density, pressure, stress etc.

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu - p\, \Delta^{\mu\nu} \tag{1}$$

- Intuitively one can understand $T^{\mu\nu}$ as the flux of 4-momentum p^μ across a surface of constant x^ν. Then,
  T^{00} is the flux of p^0 (energy) in the x^0 (time) direction, which is simply the energy density.
  $T^{i0} = T^{0i}$ is the momentum density.
  T^{ij} are momentum flux which represents the force between neighbouring infinitesimal elements of fluid.
- Fluids can never maintain exact local thermodynamic equilibrium throughout its dynamical evolution because of uncertainty principle. Also the fluid elements are in relative motion and can dissipate energy by friction. So we add the viscous stress tensor, $\Pi^{\mu\nu}$, to $T^{\mu\nu}$.

$$\Pi^{\mu\nu} = \pi^{\mu\nu} + \Delta^{\mu\nu}\Pi\,,$$

- The viscous stress tensor $\Pi^{\mu\nu}$ is split in to a part $\pi^{\mu\nu}$ (shear stress tensor) that is traceless, $\pi^\mu_\mu = 0$, and a remainder with non-vanishing trace (bulk viscous pressure).

Hydrodynamics up to second order

- The fundamental equations of viscous fluid dynamics are found by taking the appropriate projections of the conservation equations of the energy momentum tensor

$$\begin{aligned} u_\mu \partial_\mu T^{\mu\nu} &= D\epsilon + (\epsilon + p)\partial_\mu u^\mu - \Pi^{\mu\nu}\nabla_{(\mu}u_{\nu)} = 0\,, \\ \Delta^\alpha_\nu \partial_\mu T^{\mu\nu} &= (\epsilon + p)Du^\alpha - \nabla^\alpha p + \Delta^\alpha_\nu \partial_\mu \Pi^{\mu\nu} = 0\,. \end{aligned} \tag{2}$$

- The earliest formulations of relativistic hydrodynamic equations for non-ideal fluids were covariant generalisations of the Navier-Stokes equations of Newtonian non-perfect fluids. These are first-order theories and suffer instability and acausality.
- Israel-Stewart equations, the second order theory, where the viscous correction to entropy current is included. Hiscock and Lindblom showed that the perturbations in the medium evolve causally in Israel-Stewart theory around equilibrium states.

Stability and causality of third order hydrodynamics

- Israel-Stewart theory has resulted in unphysical effects such as reheating of the expanding medium and negative longitudinal pressure. In Ref. [1] a relativistic third-order evolution equation for the shear stress tensor from kinetic theory is derived.

$$\begin{aligned} \frac{\pi^{\mu\nu}}{\tau_\pi} =& -\dot{\pi}^{\langle\mu\nu\rangle} + 2\beta_\pi \sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} - \frac{4}{3}\pi^{\mu\nu}\theta + \frac{25}{7\beta_\pi}\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\pi_{\rho\gamma} \\ & - \frac{1}{3\beta_\pi}\pi_\gamma^{\langle\mu}\pi^{\nu\rangle\gamma}\theta - \frac{38}{245\beta_\pi}\pi^{\mu\nu}\pi^{\rho\gamma}\sigma_{\rho\gamma} - \frac{22}{49\beta_\pi}\pi^{\rho\langle\mu}\pi^{\nu\rangle\gamma}\sigma_{\rho\gamma} \\ & - \frac{24}{35}\nabla^{\langle\mu}\left(\pi^{\nu\rangle\gamma}\dot{u}_\gamma\tau_\pi\right) + \frac{4}{35}\nabla^{\langle\mu}\left(\tau_\pi\nabla_\gamma\pi^{\nu\rangle\gamma}\right) - \frac{2}{7}\nabla_\gamma\left(\tau_\pi\nabla^{\langle\mu}\pi^{\nu\rangle\gamma}\right) \\ & + \frac{12}{7}\nabla_\gamma\left(\tau_\pi\dot{u}^{\langle\mu}\pi^{\nu\rangle\gamma}\right) - \frac{1}{7}\nabla_\gamma\left(\tau_\pi\nabla^\gamma\pi^{\langle\mu\nu\rangle}\right) + \frac{6}{7}\nabla_\gamma\left(\tau_\pi\dot{u}^\gamma\pi^{\langle\mu\nu\rangle}\right) \\ & - \frac{2}{7}\tau_\pi\omega^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\pi_{\rho\gamma} - \frac{2}{7}\tau_\pi\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\omega_{\rho\gamma} - \frac{10}{63}\tau_\pi\pi^{\mu\nu}\theta^2 + \frac{26}{21}\tau_\pi\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma}\theta. \end{aligned} \tag{3}$$

- We perturb the energy density and fluid velocity of the system slightly that is initially in equilibrium and at rest,

$$\epsilon = \epsilon_0 + \delta\epsilon(t,x), \qquad u^\mu = (1,\vec{0}) + \delta u^\mu(t,x). \tag{4}$$

Keeping only perturbations to first order, eq. (2) becomes,

$$(\epsilon_0 + p_0)\partial_t \delta u^x + \partial_x p + \partial_\mu \delta\Pi^{\mu x} = 0. \tag{5}$$

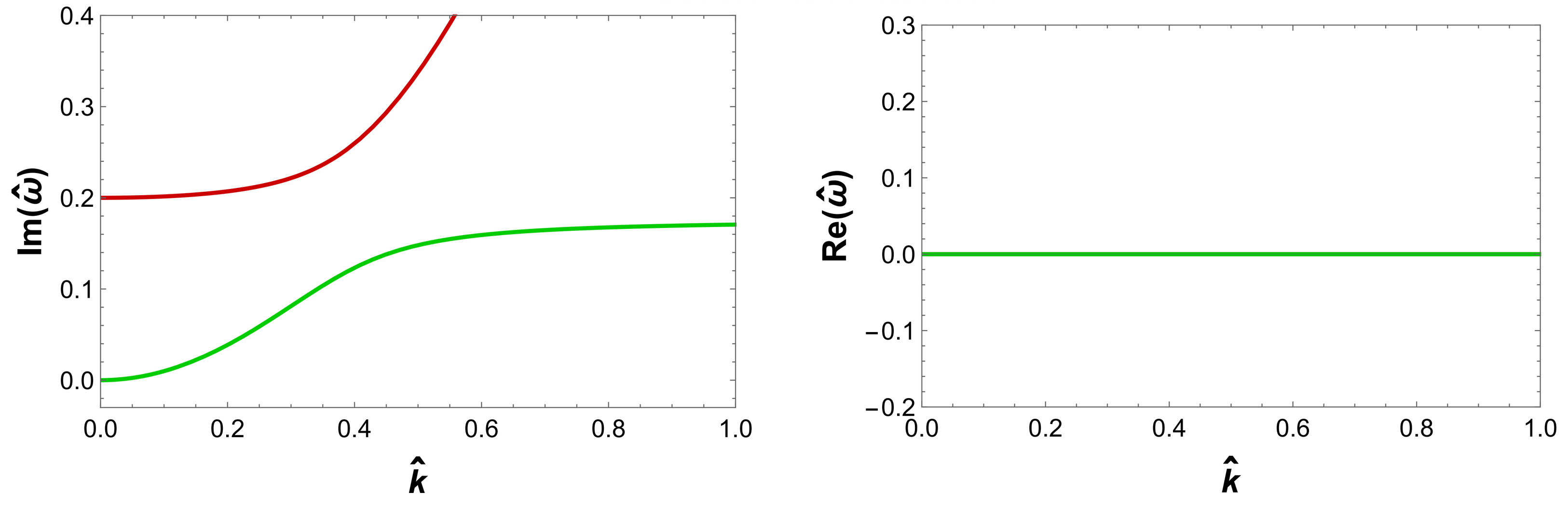
- We have to calculate the last term in the L.H.S of the above equation from eq. (3). There are total 19 terms in the R.H.S of eq. (3). So, we evaluated the contribution of each term to $\delta\Pi^{\mu x}$ individually considering only perturbations to first order.

$$\partial_\mu \delta\pi^{\mu x} = -\frac{4}{3}\eta_0 \partial_x^2 \delta u^x + \left(-\tau_\pi\partial_t + \frac{27}{105}\tau_\pi^2\partial_x^2\right)\partial_\mu\delta\pi^{\mu x}. \tag{6}$$

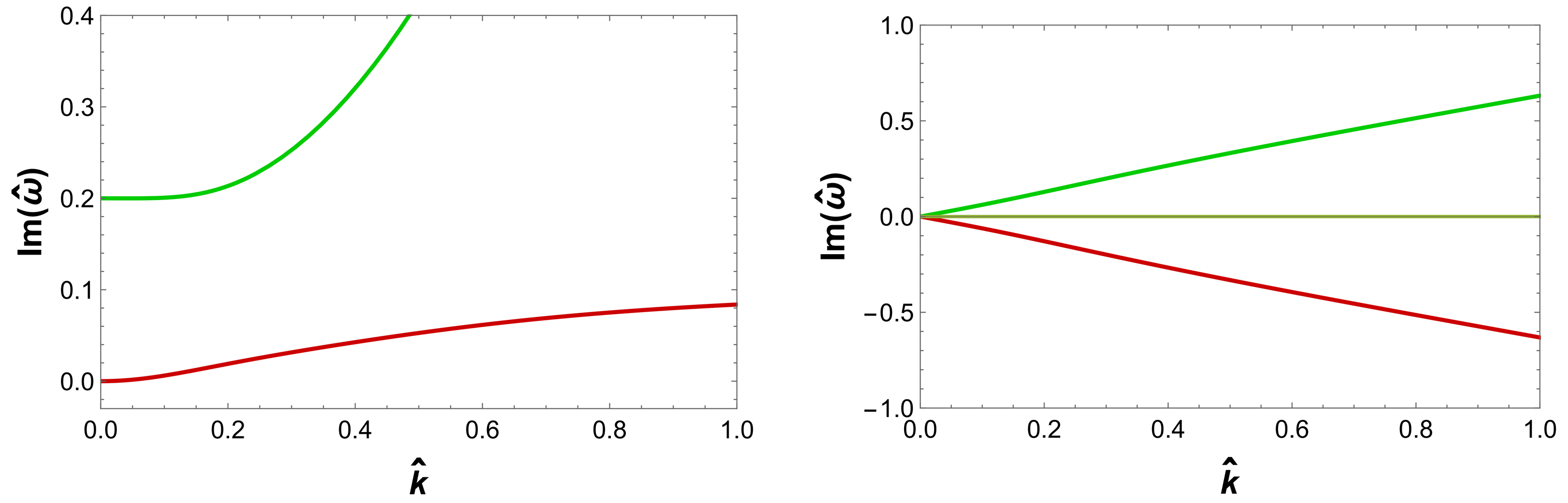
Results

Applying Fourier ansatz $\delta\epsilon = e^{i\omega t - ikx}\delta\epsilon_{\omega,k}$ and $\delta u^i = e^{i\omega t - ikx}\delta u^i_{\omega,k}$ and following the method of Romatschke Ref. [3], we obtain the dispersion relation.

Transverse modes



Longitudinal modes



SUMMARY

From the dispersion relations plotted above we can see that there exists a nonhydrodynamic mode in both transverse and longitudinal perturbation which increase as the wave number increase and doesn't saturate, thus, the theory displays acausal behaviour. The transverse modes of third order theory is purely imaginary and imaginary part of longitudinal modes is always positive, so the theory is stable.

References

[1] A. Jaiswal, Phys. Rev. C **88**, 021903(R) (2013).

[2] W. A. Hiscock and L. Lindblom, Annals Phys. **151**, 466 (1983). doi:10.1016/0003-4916(83)90288-9

[3] P. Romatschke, arXiv:0902.3663 [hep-ph].

[4] C. V. Brito and G. S. Denicol, arXiv:2107.10319 [nucl-th].