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$\mathcal{N} = 4$ supersymmetric Yang-Mills thermodynamics to order λ^2

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N = 4 supersymmetric Yang-Mills theory in 4-dimensions ($SYM_{4,4}$)

- The $SYM_{4,4}$ theory can be obtained by dimensional reduction of $SYM_{1,D}$ in D = D_{max}= 10 with all fields being in the adjoint representation of SU(N_c).

- The action and Lagrangian that generates the perturbative expansion for $SYM_{4,4}$ in Minkowski-space can be expressed as

$$S_{\mathrm{SYM}_{4,4}} = \int d^4x\, \mathcal{L}_{\mathrm{SYM}_{4,4}}\,, \qquad \text{with}$$

$$\mathcal{L}_{\mathrm{SYM}_{4,4}} = \mathrm{Tr}\Big[-\frac{1}{2}G^2_{\mu\nu} + (D_\mu\Phi_A)^2 + i\bar{\psi}_i \not{D}\psi_i - \frac{1}{2}g^2(i[\Phi_A,\Phi_B])^2$$
$$-ig\bar{\psi}_i\big[\alpha^{\mathbf{p}}_{ij}X_{\mathbf{p}} + i\beta^{\mathbf{q}}_{ij}\gamma_5 Y_{\mathbf{q}}, \psi_j\big]\Big] + \mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{gh}} + \Delta\mathcal{L}_{\mathrm{SYM}}\,,$$

- The action of N = 1 supersymmetric Yang-Mills in D dimensions ($SYM_{1,D}$) can be written in Minkowski space as

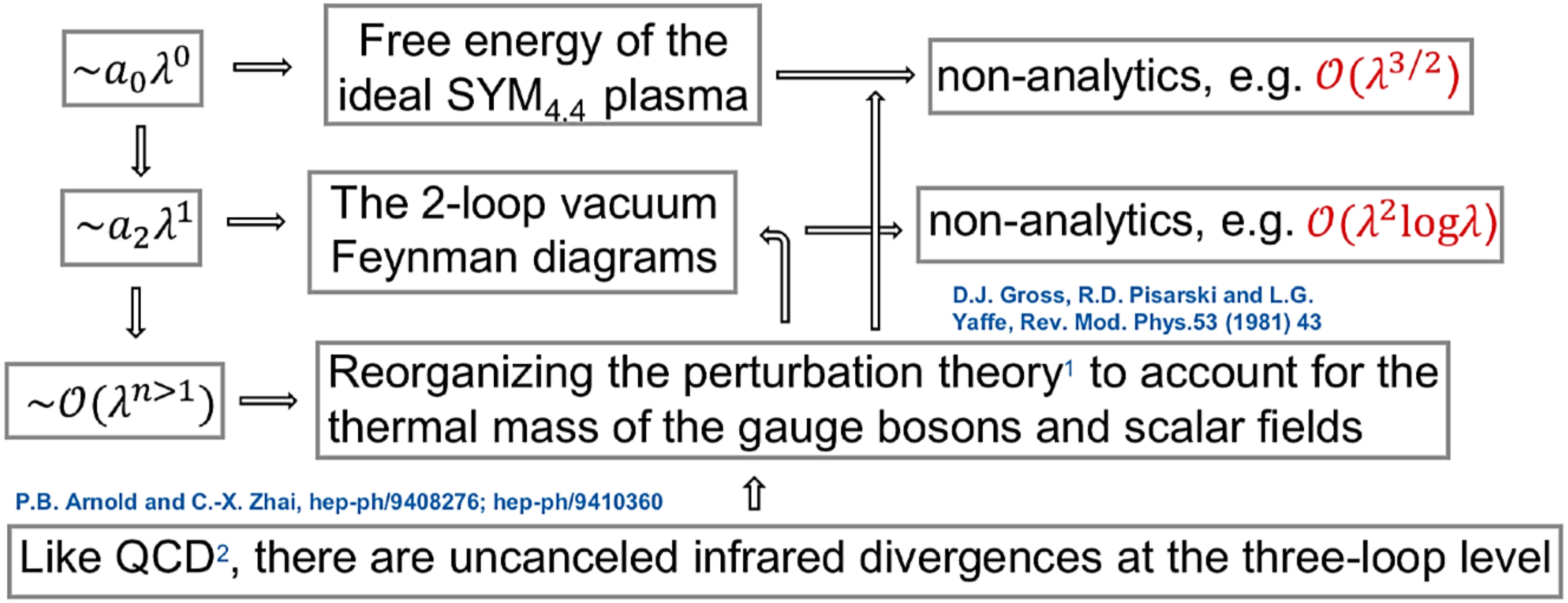
$$S_{\mathrm{SYM}_{1,\mathcal{D}}} = \int d^{\mathcal{D}}x\, \mathrm{Tr}\Big[-\frac{1}{2}G^2_{MN} + 2i\bar{\psi}\Gamma^M D_M\psi\Big]$$

Background and Motivation

- The perturbative expansion of the free energy of the $SYM_{4,4}$ at high temperature (T) can be written as in the form

$$F(\lambda \to 0) \sim T^4 \left[a_0\lambda^0 + a_2\lambda^1 + a_3\lambda^{3/2} + (a_4 + a_4'\log\lambda)\lambda^2 + \mathcal{O}\left(\lambda^{5/2}\right)\right], \quad (1)$$

where $\lambda = g^2 NC$, is the 't Hooft coupling Nc and g is the color and the coupling constant in QCD, respectivelv.



Background and Motivation

Full $\mathcal{O}(\lambda^2)$ $\Longrightarrow$ Require 3-loop calculation $\Longrightarrow$ Generate $\mathcal{O}(\lambda^{5/2})$

$\Uparrow$

Consider the dressed propagators

- In the weak-coupling limit the SYM$_{4,4}$ free energy has been calculated through $\lambda^{\frac{3}{2}}$ giving[3],

$$\frac{F}{F_{\text{ideal}}} = \frac{S}{S_{\text{ideal}}} = 1 - \frac{3}{2\pi^2}\lambda + \frac{3+\sqrt{2}}{\pi^3}\lambda^{3/2} + \cdots \quad , \quad (2)$$

A.Fotopoulos and T.R. Taylor, hep-th/9811224
M.A. Vazquez-Mozo, hep-th/9905030
C.-j. Kim and S.-J. Rey, hep-th/9905205

where: $F_{\text{ideal}} = -d_A\pi^2 T^4/6$ is the ideal or Stefan-Boltzmann limit of the free energy,

$S_{\text{ideal}} = -2d_A\pi^2 T^3/3$ is the entropy density,

$d_A = N_c^2 - 1$ is the dimension of the adjoint representation.

The aim of our work is to get 4$^{\text{th}}$ term ~$(a_4 + a_4'\log\lambda)\lambda^2$ in eq.(1)

RDR Scheme

- Conventional dimensional regularization is known to be unsuitable for supersymmetric theories *(does not manifestly preserve supersymmetry and unitarity)*

- In order to restore super-invariance, W Siegel Phys. Lett. B 84(1979) 193 proposed a new version of dimensional regularization called "regularization by dimensional reduction" (RDR) *(preserves gauge invariance, unitarity, and supersymmetry)*

- To maintain supersymmetry is to take all fields in ($SYM_{1,D}$) to be D-dimensional tensors or spinors and all momentum to be $d = D - 2\epsilon$ vectors.

- $SYM_{4,4}$ can be obtained by dimensional reduction from the $\mathcal{N}$=1 SYM theory in 10-dimension($SYM_{1,10}$) without thermal mass contributions.

- RDR has been applied to pure Yang-Mills theory; Yang-Mills theory coupled to scalars and fermions; supersymmetric QED and $\mathcal{N}$ = 1 SYM.

The resumed Lagrangian density

- The reorganized Lagrangian density in frequency space can be rewritten as

$$\mathcal{L}_{\mathrm{SYM}_{4,4}}^{\mathrm{resum}} = \{\mathcal{L}_{\mathrm{SYM}_{4,4}} + \mathrm{Tr}[m_D^2 A_0^2 \delta_{p0} - M^2 \Phi_A^2 \delta_{p0}]\} - \mathrm{Tr}[m_D^2 A_0^2 \delta_{p0} - M^2 \Phi_A^2 \delta_{p0}]$$

  where m_D is the thermal gluon mass and M is the scalar mass,

  only contribute to the zero Matsubara modes of the two fields and

  δp_0 is shorthand for the Kronecker delta function $\delta p_{0,0}$.

- Then we absorb the two A_0^2 and Φ^2 terms in the curly brackets into our unperturbed Lagrangian and treat the two terms outside the curly brackets as a perturbation.

The resumed Lagrangian density

The calculation of the thermodynamics up to $\mathcal{O}(\lambda^2)$

Feynman diagrams up to 2-loop order

$\Downarrow$

Compute directly in $\text{SYM}_{4,4}$ under the RDR scheme , since we must dress the gluon and scalar propagators differently

$\Downarrow$

$g_M^M = 4$, $\text{Tr}I_4 = 4$, the dimension for the gluon field $D = 4$ and for all momentum $d = 4 - 2\epsilon$

Feynman diagrams for the 3-loop order

3-loop gluonic and scalar counterterm with bare propagators

$\Downarrow$

Compute directly in $\text{SYM}_{4,4}$

$\Downarrow$

$g_M^M = 4$, $\text{Tr}I_4 = 4$, the dimension for the gluon field $D = 4$ and for all momentum $d = 4 - 2\epsilon$

Massless 3-loop vacuum diagrams

$\Downarrow$

Compute in $\text{SYM}_{1,10}$ under the RDR scheme

$\Downarrow$

$g_M^M = 10$, $\text{Tr}I_{32} = 8$, the dimension for the gluon field $\mathcal{D} = 10$ and for all momentum $d = 4 - 2\epsilon$

Feynman Diagram up to 3-loop order

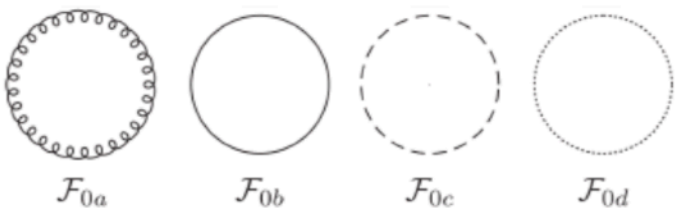


Fig.1 The 1-loop diagrams for $SYM_{4,4}$

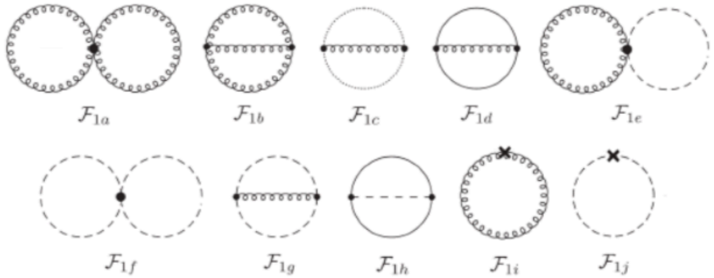


Fig.2 The 2-loop diagrams for $SYM_{4,4}$

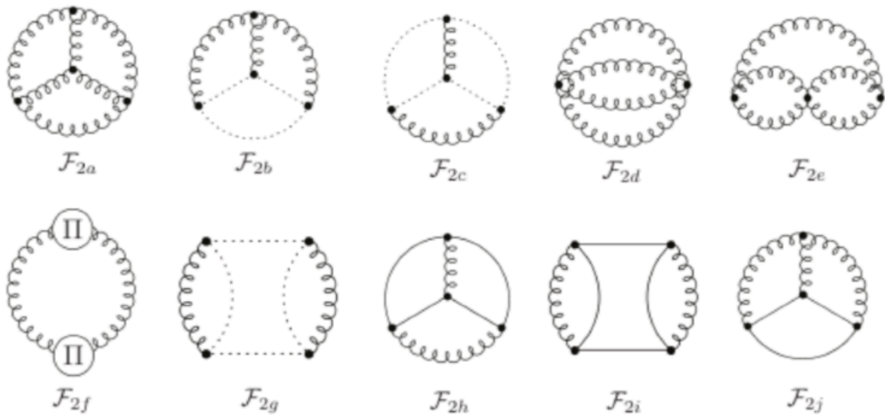


Fig.3 The 3-loop diagrams with bare propagators for $SYM_{1,10}$

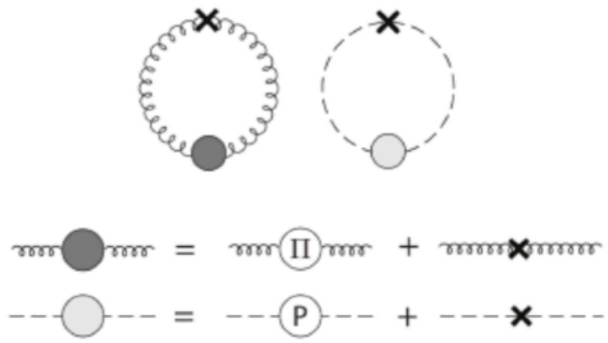


Fig.4 The 3-loop gluonic and scalar counterterm diagrams with bare propagators for $SYM_{4,4}$

The dashed lines indicate a scaler field and dotted lines indicate a ghost field. The crosses are the thermal counter terms.

Free energy up to λ^2 of $\mathrm{SYM}_{4,4}$

- The resumed one-loop free energy

$$F_{\text{1-loop}}^{\text{resum}} = d_A \mathcal{F}_{0a} + d_F \mathcal{F}_{0b} + d_S \mathcal{F}_{0c} + d_A \mathcal{F}_{0d} \ , \qquad (4)$$

with $d_F = 4d_A$ and $d_S = 6d_A$. By using the resummed gluon and scalar propagators, one obtains

$$F_{\text{1-loop}}^{\text{resum}} = d_A \left[\frac{D+4}{2} b_0 - 4 f_0 - \frac{T}{12\pi}(m_D^3 + 6M^3) \right] , \qquad (5)$$

where $b_0 \equiv \sum\!\!\!\!\!\!\!\int_P \log P^2 = -\frac{\pi^2}{45} T^4$ and $f_0 \equiv \sum\!\!\!\!\!\!\!\int_{\{P\}} \log P^2 = \frac{7\pi^2}{360} T^4$.

By imposing $D = 4,\ m_D^2 = 2\lambda T^2,\ M^2 = \lambda T^2$ and truncating at $\mathcal{O}(\epsilon^0)$, we obtain

$$F_{\text{1-loop}}^{\text{resum}} = -d_A \left(\frac{\pi^2 T^4}{6} \right) \left[1 + \frac{3+\sqrt{2}}{\pi^3} \lambda^{3/2} \right] . \qquad (6)$$

Free energy up to λ^2 of $SYM_{4,4}$

- The resumed two-loop free energy

$$F_{\text{2-loop}}^{\text{resum}} = d_A \Big\{ \lambda [\mathcal{F}_{1a} + \mathcal{F}_{1b} + \mathcal{F}_{1c} + \mathcal{F}_{1d} + \mathcal{F}_{1e} + \mathcal{F}_{1f} + \mathcal{F}_{1g} + \mathcal{F}_{1h}] + \mathcal{F}_{1i} + \mathcal{F}_{1j} \Big\} \,, \qquad (7)$$

by using the resummed gluon and scalar propagators, one obtains

$$F_{\text{2-loop}}^{\text{resum}} = \lambda d_A \Big\{ \frac{D+4}{4}\Big[(D+4)b_1^2 - 16 b_1 f_1 + 8 f_1^2\Big] + 6\frac{M^2T^2}{(4\pi)^2}\Big(\frac{3}{2} - \log\frac{M}{T} + 2\log 2\Big) + \frac{m_D^2 T^2}{(4\pi)^2}\Big(\frac{3}{4} + \frac{D}{8} - \log\frac{m_D}{T} + 2\log 2\Big) + 3\frac{m_D M T^2}{(4\pi)^2}\Big\} \,, \qquad (8)$$

where $\quad b_n \equiv \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^{2n}} \,, \qquad f_n \equiv \sum\!\!\!\!\!\!\!\int_{\{P\}} \frac{1}{P^{2n}} = (2^{2n+1-d} - 1) b_n \,, \quad n \geq 1 \,.$

By imposing $D = 4$, $m_D^2 = 2\lambda T^2$, $M^2 = \lambda T^2$ and truncating at $\mathcal{O}(\epsilon^0)$, we obtain

$$F_{\text{2-loop}}^{\text{resum}} = -d_A \left(\frac{\pi^2 T^4}{6}\right)\left[-\frac{3}{2\pi^2}\lambda - \frac{3}{2\pi^4}\left(\frac{23}{8} + \frac{3\sqrt{2}}{4} + \frac{15\log 2}{4} - \log\lambda\right)\lambda^2 \right] . \qquad (9)$$

Free energy up to λ^2 of $\mathrm{SYM}_{4,4}$

- The resumed three-loop free energy

$$F^{\text{resum}}_{\text{3-loop}} = \mathcal{F}^{\text{vacuum}}_{\text{3-loop}} + \mathcal{F}^{\text{sct}}_{\text{3-loop}} + \mathcal{F}^{\text{bct}}_{\text{3-loop}} \,, \qquad (10)$$

where
$$F^{\text{vacuum}}_{\text{3-loop}} = d_A \lambda^2 [\mathcal{F}_{2a} + \mathcal{F}_{2b} + \mathcal{F}_{2c} + \mathcal{F}_{2d} + \mathcal{F}_{2e} + \mathcal{F}_{2f} + \mathcal{F}_{2g} + \mathcal{F}_{2h} + \mathcal{F}_{2i} + \mathcal{F}_{2j}]|^{\mathcal{D}=10}_{d=4-2\epsilon} \,. \qquad (11)$$

Infrared divergences will be generated from eq.(11) due to three-momentum integrations. These divergences will be canceled by the thermal mass counterterm diagrams in Fig.4.

The shaded blob can be expressed as

$$\Delta\Pi_{\mu\nu}(P) \equiv \Pi_{\mu\nu}(P) - \Pi^{\rho\rho}(0)\delta_{\mu 0}\delta_{\nu 0}\delta_{P_0} \,,$$

and

$$\Delta\mathcal{P}(P) \equiv \mathcal{P}(P) - \mathcal{P}(0)\delta_{P_0} \,,$$

where $\Pi_{\mu\nu}(P)$ and $\mathcal{P}(P)$ are the self energy of bosons and scalars.

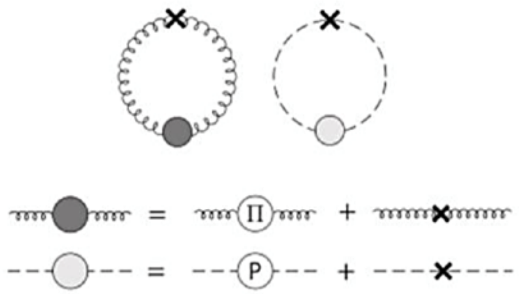


Fig.4

We obtain

$$\mathcal{F}^{\rm sct}_{\text{3-loop}} = d_A\lambda^2 6[(D+4)b_1 - 8f_1]\left[\sum\!\!\!\!\!\!\!\int_P' \frac{\Pi^b(P)}{P^2} - 2\sum\!\!\!\!\!\!\!\int_P' \frac{\Pi^f(P)}{P^2}\right] , \tag{12}$$

$$\mathcal{F}^{\rm bct}_{\text{3-loop}} = d_A\lambda^2 (D-2)[(D+4)b_1 - 8f_1]\left[\sum\!\!\!\!\!\!\!\int_P' \frac{\Pi^b(P)}{P^2} - 2\sum\!\!\!\!\!\!\!\int_P' \frac{\Pi^f(P)}{P^2} - \frac{1}{8}(D+4)\frac{T^2}{(4\pi)^2}\right] , \tag{13}$$

where $$\sum\!\!\!\!\!\!\!\int_P' \frac{\Pi^b(P)}{P^2} = \frac{T^2}{(4\pi)^2}\left[\frac{1}{4\epsilon} + \log\frac{\bar{\mu}}{4\pi T} + \log 2\pi + \frac{1}{2}\right] + \mathcal{O}(\epsilon) ,$$

$$\sum\!\!\!\!\!\!\!\int_P' \frac{\Pi^f(P)}{P^2} = \frac{T^2}{(4\pi)^2}\log 2 + \mathcal{O}(\epsilon) .$$

By imposing $D = 4, \mathrm{Tr} I_{32} = 8, m_D^2 = 2\lambda T^2,\ M^2 = \lambda T^2$ and truncating at $\mathcal{O}(\epsilon^0)$, we obtain

$$F^{\rm resum}_{\text{3-loop}} = -d_A\left(\frac{\pi^2 T^4}{6}\right)\frac{\lambda^2}{2\pi^4}\left[3 + 3\gamma + 3\frac{\zeta'(-1)}{\zeta(-1)} + 5\log 2 - 6\log\pi\right] . \tag{14}$$

Free energy up to λ^2 of $SYM_{4,4}$ – Result

- By combining eqns. (6) (9) and (14), the result of the resumed free energy up to 3 loop level for $SYM_{4,4}$ under the RDR scheme is

$$F = -d_A\left(\frac{\pi^2 T^4}{6}\right)\left\{1 - \frac{3}{2\pi^2}\lambda + \frac{3+\sqrt{2}}{\pi^3}\lambda^{\frac{3}{2}} + \frac{1}{\pi^4}\left[-\frac{45}{16} - \frac{9\sqrt{2}}{8} + \frac{3}{2}\gamma_E + \frac{3}{2}\frac{\zeta'(-1)}{\zeta(-1)} - \frac{25}{8}\log 2 - 3\log\pi + \frac{3}{2}\log\lambda\right]\lambda^2\right\}. \quad (15)$$

- This result holds for all N_c .
- The result (15) is manifestly finite due to an explicit cancellation between three-loop infrared singularities and the three-loop counter term diagrams.
- These cancellations remove all infrared divergent contributions.
- In addition, there are no remaining poles due to ultraviolet divergences, since the coupling does not run in $SYM_{4,4}$ and, hence, no coupling constant renormalization counter term is required.

Large – N_c generalized Padé approximant

- With new perturbative coefficients in hand one can produce an updated Padé approximant.
  J.P. Blaizot, E.Iancu, U.Kraemmer and A.Rebhan, hep-ph/0611393

- Based on the large-Nc structure of the strong-coupling expansion, we find that the following form can reconstruct all known coefficients in both the weak- and strong-coupling limits

$$\frac{S}{S_{\mathrm{ideal}}} = \frac{1 + a\lambda^{1/2} + b\lambda + c\lambda^{3/2} + d\lambda^2 + e\lambda^{5/2}}{1 + a\lambda^{1/2} + \bar{b}\lambda + \frac{4}{3}c\lambda^{3/2} + \frac{4}{3}d\lambda^2 + \frac{4}{3}e\lambda^{5/2}}, \quad (16)$$

with

$$a = \frac{4\pi^2}{135\zeta(3)} + \frac{2(3+\sqrt{2})}{3\pi},$$

$$b = \frac{1}{\pi^2}\log\left(\frac{\lambda}{\pi^2}\right) + \frac{16\pi[45(3+\sqrt{2})\zeta(3)+\pi^3]}{18225\zeta^2(3)} + \frac{72\left(\gamma_E + \frac{\zeta'(-1)}{\zeta(-1)}\right) + 138\sqrt{2} + 109 - 150\log(2)}{72\pi^2},$$

$$\bar{b} = b + \frac{3}{2\pi^2}, \qquad c = \frac{2}{15\zeta(3)}, \qquad d = \frac{180(3+\sqrt{2})\zeta(3) + 8\pi^3}{2025\pi\zeta^2(3)}, \qquad e = \frac{2b}{15\zeta(3)} - \frac{3}{5\pi^2\zeta(3)}.$$

Scaled entropy density as a function of λ

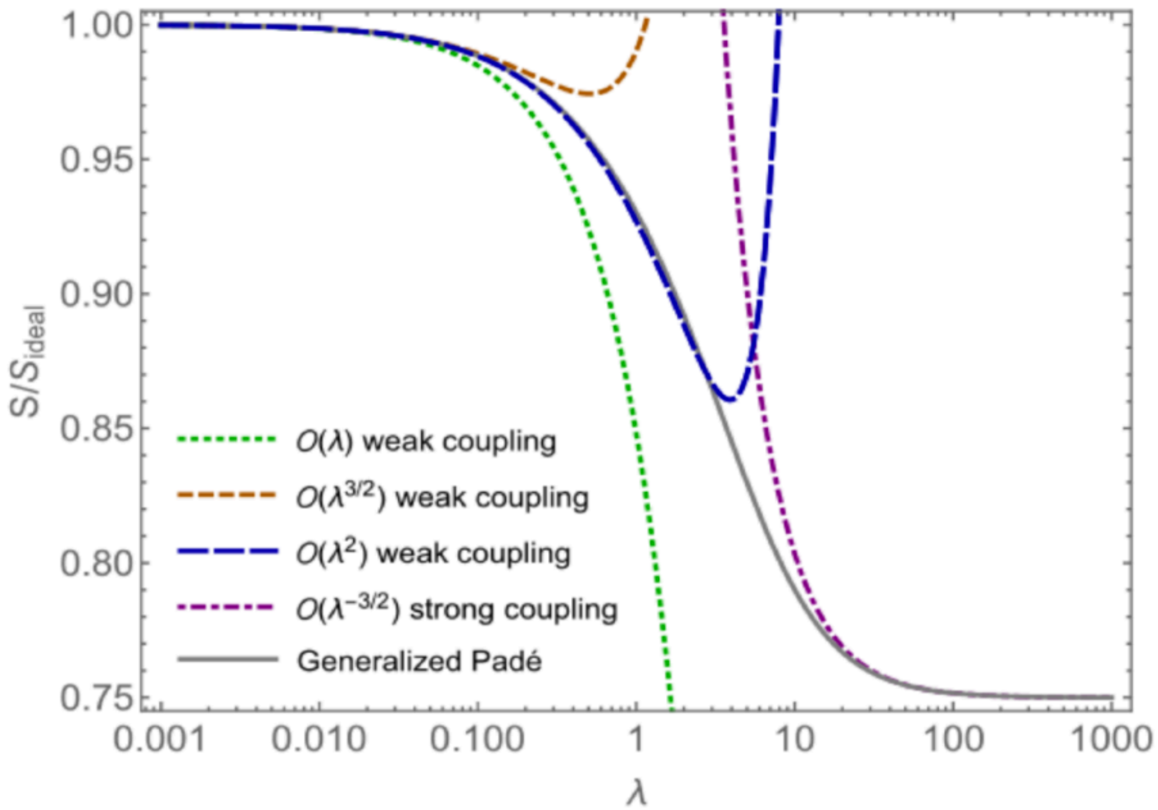


Fig.5 $\mathrm{SYM}_{4,4}$ scaled entropy density $\mathcal{S}/\mathcal{S}_{\mathrm{ideal}}$ as a function of λ. The green dotted, red dashed, and blue long-dashed curves correspond to the perturbative result truncated at $\mathcal{O}(\lambda)$, $\mathcal{O}(\lambda^{3/2})$, and $\mathcal{O}(\lambda^2)$, respectively. The purple dot-dashed curve corresponds to the large-N_c strong-coupling result truncated at $O(\lambda^{-3/2})$. The solid gray line is the updated Padé approximant.

Comparison of entropy density

- One can take the value of λ at which the truncated perturbative solutions significantly depart from the Padé approximant as an estimate of the range of validity of each perturbative truncation.

- From Fig. 5, when truncated at $\mathcal{O}(\lambda)$, one must have $\lambda \lesssim 0.02$. At $\mathcal{O}(\lambda^{3/2})$, one finds $\lambda \lesssim 0.2$, and at $\mathcal{O}(\lambda^2)$, one finds $\lambda \lesssim 2$.

- In comparison to the convergence of the perturbative QCD free energy we observe that the $O(\lambda^2)$ truncation in $SYM_{4,4}$ has $\mathcal{P}/\mathcal{P}_{\text{ideal}} = \mathcal{S}/\mathcal{S}_{\text{ideal}} < 1$ for $\lambda \lesssim 10$, whereas the $O(\lambda^2)$ truncation in QCD has $\mathcal{P} > \mathcal{P}_{\text{ideal}}$ for $\lambda \geq 3.5$.

- In contrast, lattice QCD measurements of the pressure find $\mathcal{P} < \mathcal{P}_{\text{ideal}}$.

- Suggestion: perturbative expansion of the $SYM_{4,4}$ free energy might have better convergence than the perturbative expansion of the QCD free energy.

Conclusions and Outlook

- We computed the thermodynamic function of $SYM_{4,4}$ to $\mathcal{O}(\lambda^2)$ under RDR Scheme.
- Having computed new coefficients, we then constructed a large-N_c Padé approximant that interpolates between the weak- and strong-coupling limits.
- In the near future we plan to also compute the coefficient of $\lambda^{5/2}$ in the $SYM_{4,4}$ free energy.
- We also plan to pursue a three-loop HTLpt calculation of $SYM_{4,4}$ thermodynamics to extend our previous two-loop HTLpt calculation.