# Helicity Evolution at Small $x$: The Single-Logarithmic Contribution

Yossathorn (Josh) Tawabutr
with Yuri Kovchegov and Andrey Tarasov

The Ohio State University

*tawabutr.1@osu.edu*

September 5, 2021

Based on: 2005.07285 and 2104.11765

# Background

Proton helicity can be decomposed into spin and orbital angular momentum (OAM) of quarks and gluons [Jaffe and Manohar, 1990]

$$\frac{1}{2} = S_q + S_G + L_q + L_G \tag{1}$$

where

$$S_q(Q^2) = \frac{1}{2}\int_0^1 dx\, \Delta\Sigma(x, Q^2) = \frac{1}{2}\int_0^1 dx \sum_f \left[\Delta q_f(x, Q^2) + \Delta\overline{q}_f(x, Q^2)\right]. \tag{2}$$

Experiments have measured $S_q$ but can only include $0 < x_{\text{min}} \leq x \leq 1$.

**Objective:** Find the contribution to $S_q$ coming from $\Delta\Sigma$ as $x \to 0$.

# Background

At small Bjorken-$x$, quark helicity distribution satisfies

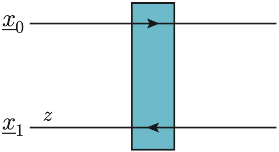
$$\Delta\Sigma(x, Q^2) = \frac{N_c N_f}{2\pi^3} \int_{\Lambda^2/s}^{1} \frac{dz}{z} \int_{1/zs}^{1/zQ^2} \frac{dx_{10}^2}{x_{10}^2} \int d^2\underline{b}\; Q(\underline{x}_1, \underline{x}_0, z), \qquad (3)$$

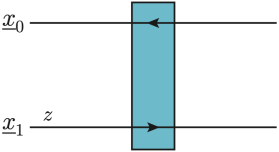
where $\underline{b} = \frac{\underline{x}_0 + \underline{x}_1}{2}$ and $\underline{x}_{10} = \underline{x}_1 - \underline{x}_0$.

Here, $Q(\underline{x}_1, \underline{x}_0, z) \equiv Q_{10}(z)$ is the quark (longitudinally) polarized dipole amplitude.

# Quark Unpolarized Dipole Amplitude

$S_{10}(z)$ corresponds to a minus-moving quark dipole interacting with a plus-moving target proton, represented by the blue rectangle.

$S_{10}(z) =$ 

$+$ 

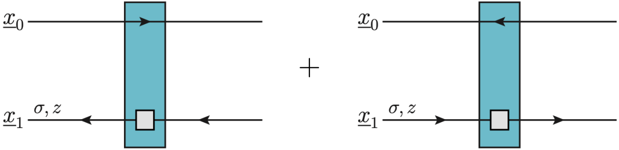

$$= \frac{1}{2N_c} \, \mathrm{Re} \left[ \left\langle \mathcal{T} \, \mathrm{tr} \left[ V_{\underline{0}} V_{\underline{1}}^{\dagger} \right] \right\rangle (z) + \left\langle \mathcal{T} \, \mathrm{tr} \left[ V_{\underline{1}} V_{\underline{0}}^{\dagger} \right] \right\rangle (z) \right] . \qquad (4)$$

- $V_{\underline{i}}$ is the fundamental Wilson's line at $\underline{x}_i$.
- The angle brackets average over target proton's wave function.

# Quark Polarized Dipole Amplitude

Diagrammatically, $Q_{10}(z)$ corresponds to a quark dipole, one of which has helicity $\sigma$, interacting with a polarized target proton (blue rectangle).

$$Q_{10}(z) =$$



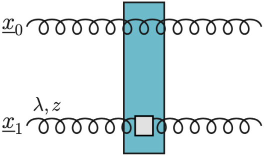

$$= \frac{zs}{2N_c} \, \mathrm{Re} \left[ \left\langle \mathcal{T} \, \mathrm{tr} \left[ V_{\underline{0}} V_{\underline{1}}^{\mathrm{pol}\,\dagger} \right] \right\rangle (z) + \left\langle \mathcal{T} \, \mathrm{tr} \left[ V_{\underline{1}}^{\mathrm{pol}} V_{\underline{0}}^{\dagger} \right] \right\rangle (z) \right]. \quad (5)$$

- $V_{\underline{0}}$ is the fundamental unpolarized Wilson's line at $\underline{x}_0$.
- $V_{\underline{1}}^{\mathrm{pol}}$ is the fundamental *polarized Wilson's line* at $\underline{x}_1$.
- The angle brackets average over target proton's wave function.
- At Born level, $Q_{10}(z) \sim 1$ because the helicity-dependent tree-level cross-section $\sim \frac{1}{zs}$.

# Gluon Polarized Dipole Amplitude

Similar to quark, consider a gluon dipole, one of which has helicity $\lambda$, interacting with a polarized target proton (blue rectangle).

$G_{10}(z) =$

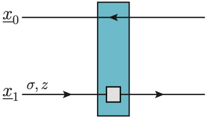



$$= \frac{zs}{2(N_c^2 - 1)} \, \mathrm{Re} \left[ \left\langle \mathcal{T} \, \mathrm{tr} \left[ U_{\underline{0}} U_{\underline{1}}^{\mathrm{pol}\,\dagger} \right] \right\rangle (z) + \left\langle \mathcal{T} \, \mathrm{tr} \left[ U_{\underline{1}}^{\mathrm{pol}} U_{\underline{0}}^{\dagger} \right] \right\rangle (z) \right]. \quad (6)$$

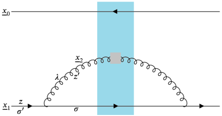
- $U_{\underline{0}}$ is the adjoint unpolarized Wilson's line at $\underline{x}_0$.
- $U_{\underline{1}}^{\mathrm{pol}}$ is the adjoint *polarized Wilson's line* at $\underline{x}_1$.
- The angle brackets average over target proton's wave function.

# Evolution

- The polarized dipole amplitudes obey integral equations resulting from quark/gluon splitting outside the target shockwave:




$\rightarrow$



$+$ (other diagrams)

- To the first order in $\alpha_s$, both dipole amplitudes evolve as

$$\alpha_s \left[ \underbrace{\int \frac{dz'}{z'} \int \frac{dx_{21}^2}{x_{21}^2}}_{\text{Double-logarithmic}} + \underbrace{\int dz' \int \frac{dx_{32}^2}{x_{32}^2} + \int \frac{dz'}{z'} \int dx_{21}^2}_{\text{Single-logarithmic}} + \dots \right]$$

$$\times \text{(dipole amplitudes)}\,.$$

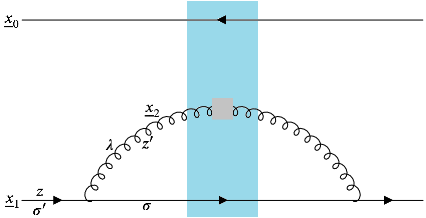
- The single-logarithmic (SLA) terms (resumming $\alpha_s \ln \frac{1}{x}$) are subleading to the double-logarithmic (DLA) term (resumming $\alpha_s \ln^2 \frac{1}{x}$).

# Longitudinally Soft Parton Emission

- In a splitting in the limit $z' \ll z$, the longitudinal $z'$-integral is logarithmic, giving the evolution terms

$$\alpha_s \int \frac{dz'}{z'} \Bigg[ \underbrace{A \int \frac{dx_{21}^2}{x_{21}^2}}_{\text{DLA}} + \underbrace{B \int dx_{21}^2}_{\text{SLA}_\text{L}} \Bigg] \text{(dipole amps)} \qquad (7)$$

- $\text{SLA}_L$: single-logarithmic term with logarithmic longitudinal integral.

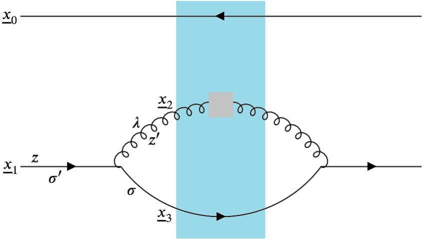


# Longitudinally Hard Parton Emission

- In the limit $z' \sim z$, a parton splitting yields the SLA$_T$ terms, i.e. single-logarithmic terms with logarithmic transverse integral

$$\alpha_s \int dz' \, \Delta P(z'/z) \int \frac{dx_{32}^2}{x_{32}^2} \, \text{(dipole amps)} \tag{8}$$

- Here, $\Delta P(z'/z)$ is the polarized DGLAP splitting function.
- Since $z' \sim (z - z') \sim z$, neither $\underline{x}_3$ nor $\underline{x}_2$ is close to $\underline{x}_1$, but we still have $x_{32} \ll x_{10}$.



# Full Evolution Equations

Including both soft and hard parton emissions, we have up to SLA:

$$
\begin{aligned}
&\frac{1}{N_c}\Big\langle\!\!\Big\langle \mathrm{tr}\Big[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\Big]\Big\rangle\!\!\Big\rangle(z_{\min}, z_{\mathrm{pol}}) = \frac{1}{N_c}\Big\langle\!\!\Big\langle \mathrm{tr}\Big[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\Big]\Big\rangle\!\!\Big\rangle_0(z_{\mathrm{pol}}) + \frac{1}{2\pi^2}\int\limits_{\Lambda^2/s}^{z_{\min}}\frac{dz'}{z'}\int\limits_{1/(z's)} d^2x_2 \\
&\quad\times\Bigg[\Bigg(\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\,\theta(x_{10}^2 z_{\min} - x_{21}^2 z') - \alpha_s(\min\{1/x_{21}^2, 1/x_{20}^2\})\,\frac{\underline{x}_{21}\cdot\underline{x}_{20}}{x_{21}^2\, x_{20}^2}\,\theta(x_{10}^2 z_{\min} - \max\{x_{21}^2, x_{20}^2\} z')\Bigg) \\
&\qquad\times\frac{2}{N_c}\Big\langle\!\!\Big\langle \mathrm{tr}\Big[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{1}}^{\dagger}\Big]\, U_{\underline{2}}^{\mathrm{pol}\,ba}\Big\rangle\!\!\Big\rangle(z', z') \\
&\qquad+\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\,\theta(x_{10}^2 z_{\min} - x_{21}^2 z')\,\frac{1}{N_c}\Big\langle\!\!\Big\langle \mathrm{tr}\Big[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{2}}^{\mathrm{pol}\,\dagger}\Big]\, U_{\underline{1}}^{ba}\Big\rangle\!\!\Big\rangle(z', z')\Bigg] \\
&+\frac{1}{2\pi^2}\int\limits_{\Lambda^2/s}^{z_{\min}}\frac{dz'}{z'}\int\limits_{1/(z's)} d^2x_2\, K_{\mathrm{rcBK}}(\underline{x}_0, \underline{x}_1; \underline{x}_2)\,\theta(x_{10}^2 z_{\min} - x_{21}^2 z') \\
&\quad\times\frac{1}{N_c}\Big[\Big\langle\!\!\Big\langle \mathrm{tr}\Big[V_{\underline{0}}\, V_{\underline{2}}^{\dagger}\Big]\,\mathrm{tr}\Big[V_{\underline{2}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\Big]\Big\rangle\!\!\Big\rangle(z', z_{\mathrm{pol}}) - N_c\Big\langle\!\!\Big\langle \mathrm{tr}\Big[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\Big]\Big\rangle\!\!\Big\rangle(z', z_{\mathrm{pol}})\Big] \\
&-\frac{1}{2\pi^2}\int\limits_0^{z_{\mathrm{pol}}}\frac{dz'}{z_{\mathrm{pol}}}\int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}}\frac{d^2x_{32}}{x_{32}^2}\,\theta(x_{10}^2 z_{\min} z_{\mathrm{pol}} - x_{32}^2 z'(z_{\mathrm{pol}} - z'))\,\alpha_s\Big(\frac{1}{x_{32}^2}\Big) \\
&\quad\times\Bigg[\frac{1}{N_c}\Big\langle\!\!\Big\langle \mathrm{tr}\Big[t^b\, V_{\underline{0}}\, t^a\, V^{\dagger}_{\underline{x}_1 - \frac{z'}{z_{\mathrm{pol}}}\underline{x}_{32}}\Big]\, U^{\mathrm{pol}\,ba}_{\underline{x}_1 + \left(1 - \frac{z'}{z_{\mathrm{pol}}}\right)\underline{x}_{32}}\Big\rangle\!\!\Big\rangle(\min\{z_{\min}, z', z_{\mathrm{pol}} - z'\}, z') \\
&\qquad+\frac{1}{N_c}\Big\langle\!\!\Big\langle \mathrm{tr}\Big[t^b\, V_{\underline{0}}\, t^a\, V^{\mathrm{pol}\,\dagger}_{\underline{x}_1 + \left(1 - \frac{z'}{z_{\mathrm{pol}}}\right)\underline{x}_{32}}\Big]\, U^{ba}_{\underline{x}_1 - \frac{z'}{z_{\mathrm{pol}}}\underline{x}_{32}}\Big\rangle\!\!\Big\rangle(\min\{z_{\min}, z', z_{\mathrm{pol}} - z'\}, z')\Bigg] \\
&+\frac{C_F}{2\pi^2}\int\limits_0^{z_{\mathrm{pol}}}\frac{dz'}{z_{\mathrm{pol}}}\Big(1 + \frac{z'}{z_{\mathrm{pol}}}\Big)\int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}}\frac{d^2x_{32}}{x_{32}^2}\,\theta(x_{10}^2 z_{\min} z_{\mathrm{pol}} - x_{32}^2 z'(z_{\mathrm{pol}} - z'))\,\alpha_s\Big(\frac{1}{x_{32}^2}\Big) \\
&\quad\times\frac{1}{N_c}\Big\langle\!\!\Big\langle \mathrm{tr}\Big[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\Big]\Big\rangle\!\!\Big\rangle(\min\{z_{\min}, z', z_{\mathrm{pol}} - z'\}, z_{\mathrm{pol}}).
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{N_c^2-1}\Big\langle\!\!\Big\langle \mathrm{Tr}\Big[U_{\underline{0}}\, U_{\underline{1}}^{\mathrm{pol}\,\dagger}\Big]\Big\rangle\!\!\Big\rangle(z_{\min}, z_{\mathrm{pol}}) = \frac{1}{N_c^2-1}\Big\langle\!\!\Big\langle \mathrm{Tr}\Big[U_{\underline{0}}\, U_{\underline{1}}^{\mathrm{pol}\,\dagger}\Big]\Big\rangle\!\!\Big\rangle_0(z_{\mathrm{pol}}) + \frac{1}{2\pi^2}\int\limits_{\Lambda^2/s}^{z_{\min}}\frac{dz'}{z'}\int\limits_{1/(z's)} d^2x_2 \\
&\quad\times\Bigg[\Bigg(\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\,\theta(x_{10}^2 z_{\min} - x_{21}^2 z') - \alpha_s(\min\{1/x_{21}^2, 1/x_{20}^2\})\,\frac{\underline{x}_{21}\cdot\underline{x}_{20}}{x_{21}^2\, x_{20}^2}\,\theta(x_{10}^2 z_{\min} - \max\{x_{21}^2, x_{20}^2\} z')\Bigg) \\
&\qquad\times\frac{4}{N_c^2-1}\Big\langle\!\!\Big\langle \mathrm{Tr}\Big[T^b\, U_{\underline{0}}\, T^a\, U_{\underline{1}}^{\dagger}\Big]\, U_{\underline{2}}^{\mathrm{pol}\,ba}\Big\rangle\!\!\Big\rangle(z', z') \\
&\qquad-\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\,\theta(x_{10}^2 z_{\min} - x_{21}^2 z')\,\frac{N_f}{N_c^2-1}\Big\langle\!\!\Big\langle \mathrm{tr}\Big[t^b\, V_{\underline{1}}\, t^a\, V_{\underline{2}}^{\mathrm{pol}\,\dagger}\Big]\, U_{\underline{0}}^{ba} + \mathrm{tr}\Big[t^b\, V_{\underline{2}}^{\mathrm{pol}}\, t^a\, V_{\underline{1}}^{\dagger}\Big]\, U_{\underline{0}}^{ba}\Big\rangle\!\!\Big\rangle(z', z')\Bigg] \\
&+\frac{1}{\pi^2}\int\limits_{\Lambda^2/s}^{z_{\min}}\frac{dz'}{z'}\int\limits_{1/(z's)} d^2x_2\, K_{\mathrm{rcBK}}(\underline{x}_0, \underline{x}_1; \underline{x}_2)\,\theta(x_{10}^2 z_{\min} - x_{21}^2 z') \\
&\quad\times\frac{1}{N_c^2-1}\Big[\Big\langle\!\!\Big\langle \mathrm{Tr}\Big[T^b\, U_{\underline{0}}\, T^a\, U_{\underline{1}}^{\mathrm{pol}\,\dagger}\Big]\, U_{\underline{2}}^{ba}\Big\rangle\!\!\Big\rangle(z', z_{\mathrm{pol}}) - N_c\Big\langle\!\!\Big\langle \mathrm{Tr}\Big[U_{\underline{0}}\, U_{\underline{1}}^{\mathrm{pol}\,\dagger}\Big]\Big\rangle\!\!\Big\rangle(z', z_{\mathrm{pol}})\Big] \\
&-\frac{1}{2\pi^2}\int\limits_0^{z_{\mathrm{pol}}}\frac{dz'}{z_{\mathrm{pol}}}\int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}}\frac{d^2x_{32}}{x_{32}^2}\,\theta(x_{10}^2 z_{\min} z_{\mathrm{pol}} - x_{32}^2 z'(z_{\mathrm{pol}} - z'))\,\alpha_s\Big(\frac{1}{x_{32}^2}\Big) \\
&\quad\times\frac{4}{N_c^2-1}\Big\langle\!\!\Big\langle \mathrm{Tr}\Big[T^b\, U_{\underline{0}}\, T^a\, U^{\dagger}_{\underline{x}_1 - \frac{z'}{z_{\mathrm{pol}}}\underline{x}_{32}}\Big]\, U^{\mathrm{pol}\,ba}_{\underline{x}_1 + \left(1 - \frac{z'}{z_{\mathrm{pol}}}\right)\underline{x}_{32}}\Big\rangle\!\!\Big\rangle(\min\{z_{\min}, z', z_{\mathrm{pol}} - z'\}, z') \\
&+\frac{1}{\pi^2}\int\limits_0^{z_{\mathrm{pol}}}\frac{dz'}{z_{\mathrm{pol}}}\int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}}\frac{d^2x_{32}}{x_{32}^2}\,\theta(x_{10}^2 z_{\min} z_{\mathrm{pol}} - x_{32}^2 z'(z_{\mathrm{pol}} - z'))\,\alpha_s\Big(\frac{1}{x_{32}^2}\Big)\,\frac{N_f}{N_c^2-1} \\
&\quad\times\Big\langle\!\!\Big\langle \mathrm{tr}\Big[t^b\, V_{\underline{x}_1 - \frac{z'}{z_{\mathrm{pol}}}\underline{x}_{32}}\, t^a\, V^{\mathrm{pol}\,\dagger}_{\underline{x}_1 + \left(1 - \frac{z'}{z_{\mathrm{pol}}}\right)\underline{x}_{32}}\Big]\, U_{\underline{0}}^{ba} \\
&\qquad+\mathrm{tr}\Big[t^b\, V^{\mathrm{pol}}_{\underline{x}_1 + \left(1 - \frac{z'}{z_{\mathrm{pol}}}\right)\underline{x}_{32}}\, t^a\, V^{\dagger}_{\underline{x}_1 - \frac{z'}{z_{\mathrm{pol}}}\underline{x}_{32}}\Big]\, U_{\underline{0}}^{ba}\Big\rangle\!\!\Big\rangle(\min\{z_{\min}, z', z_{\mathrm{pol}} - z'\}, z') \\
&+\frac{1}{2\pi^2}\int\limits_0^{z_{\mathrm{pol}}}\frac{dz'}{z_{\mathrm{pol}}}\int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}}\frac{d^2x_{32}}{x_{32}^2}\Bigg[N_c\Big(2 - \frac{z'}{z_{\mathrm{pol}}} + \frac{z'^2}{z_{\mathrm{pol}}^2}\Big) - \frac{N_f}{2}\Big(\frac{z'^2}{z_{\mathrm{pol}}^2} + \Big(1 - \frac{z'}{z_{\mathrm{pol}}}\Big)^2\Big)\Bigg] \\
&\quad\times\theta\left(x_{10}^2 z_{\min} z_{\mathrm{pol}} - x_{32}^2 z'(z_{\mathrm{pol}} - z')\right)\alpha_s\Big(\frac{1}{x_{32}^2}\Big)\,\frac{1}{N_c^2-1}\Big\langle\!\!\Big\langle \mathrm{Tr}\Big[U_{\underline{0}}\, U_{\underline{1}}^{\mathrm{pol}\,\dagger}\Big]\Big\rangle\!\!\Big\rangle(\min\{z_{\min}, z', z_{\mathrm{pol}} - z'\}, z_{\mathrm{pol}}).
\end{aligned}
$$

- $\mathrm{SLA}_T$ terms are written in blue.
- The running coupling correction is SLA and also has to be included.

# Full Evolution Equations

Including both double-log (DLA) and single-log (SLA) terms, we have

$$
\begin{aligned}
&\frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle(z_{\min}, z_{\mathrm{pol}}) = \frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle_0(z_{\mathrm{pol}}) + \frac{1}{2\pi^2}\int\limits_{\Lambda^2/s}^{z_{\min}} \frac{dz'}{z'} \int\limits_{1/(z's)} d^2 x_2 \\
&\quad\times \Bigg[\left(\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\,\theta(x_{10}^2 z_{\min} - x_{21}^2 z') - \alpha_s(\min\{1/x_{21}^2, 1/x_{20}^2\})\,\frac{\underline{x}_{21}\cdot\underline{x}_{20}}{x_{21}^2\, x_{20}^2}\,\theta(x_{10}^2 z_{\min} - \max\{x_{21}^2, x_{20}^2\} z')\right) \\
&\qquad\times \frac{2}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{1}}^{\dagger}\right] U_{\underline{2}}^{\mathrm{pol}\, ba}\right\rangle\!\!\right\rangle(z', z') \\
&\qquad+ \frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\,\theta(x_{10}^2 z_{\min} - x_{21}^2 z')\,\frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{2}}^{\mathrm{pol}\,\dagger}\right] U_{\underline{1}}^{ba}\right\rangle\!\!\right\rangle(z', z')\Bigg] \\
&+ \frac{1}{2\pi^2}\int\limits_{\Lambda^2/s}^{z_{\min}} \frac{dz'}{z'} \int\limits_{1/(z's)} d^2 x_2\, K_{\mathrm{rcBK}}(\underline{x}_0, \underline{x}_1; \underline{x}_2)\,\theta(x_{10}^2 z_{\min} - x_{21}^2 z') \\
&\quad\times \frac{1}{N_c}\left[\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{2}}^{\dagger}\right] \mathrm{tr}\left[V_{\underline{2}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle(z', z_{\mathrm{pol}}) - N_c \left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle(z', z_{\mathrm{pol}})\right] \\
&- \frac{1}{2\pi^2}\int\limits_0^{z_{\mathrm{pol}}} \frac{dz'}{z_{\mathrm{pol}}} \int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}} \frac{d^2 x_{32}}{x_{32}^2}\,\theta(x_{10}^2 z_{\min} z_{\mathrm{pol}} - x_{32}^2 z'(z_{\mathrm{pol}} - z'))\,\alpha_s\left(\frac{1}{x_{32}^2}\right) \\
&\quad\times \Bigg[\frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V^{\dagger}_{\underline{x}_1 - \frac{z'}{z_{\mathrm{pol}}}\underline{x}_{32}}\right] U^{\mathrm{pol}\, ba}_{\underline{x}_1 + \left(1 - \frac{z'}{z_{\mathrm{pol}}}\right)\underline{x}_{32}}\right\rangle\!\!\right\rangle(\min\{z_{\min}, z', z_{\mathrm{pol}} - z'\}, z') \\
&\qquad+ \frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V^{\mathrm{pol}\,\dagger}_{\underline{x}_1 + \left(1 - \frac{z'}{z_{\mathrm{pol}}}\right)\underline{x}_{32}}\right] U^{ba}_{\underline{x}_1 - \frac{z'}{z_{\mathrm{pol}}}\underline{x}_{32}}\right\rangle\!\!\right\rangle(\min\{z_{\min}, z', z_{\mathrm{pol}} - z'\}, z')\Bigg] \\
&+ \frac{C_F}{2\pi^2}\int\limits_0^{z_{\mathrm{pol}}} \frac{dz'}{z_{\mathrm{pol}}}\left(1 + \frac{z'}{z_{\mathrm{pol}}}\right) \int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}} \frac{d^2 x_{32}}{x_{32}^2}\,\theta(x_{10}^2 z_{\min} z_{\mathrm{pol}} - x_{32}^2 z'(z_{\mathrm{pol}} - z'))\,\alpha_s\left(\frac{1}{x_{32}^2}\right) \\
&\quad\times \frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle(\min\{z_{\min}, z', z_{\mathrm{pol}} - z'\}, z_{\mathrm{pol}}).
\end{aligned}
$$

- $\mathrm{SLA}_T$ terms are written in blue.
- The running coupling correction is SLA and also has to be included.

# Closed Evolution Equations

- At large $N_c$ and large $N_c\&N_f$, the equations become closed but *non-linear* because the evolution of unpolarized dipoles are also SLA.
- For example, at large $N_c$, (half of) the equations are:

$$
\begin{aligned}
G_{10}(z_{\min}, z_{\text{pol}}) &= G^{(0)}_{10}(z_{\text{pol}}) + \frac{N_c}{\pi^2} \int\limits_{\Lambda^2/s}^{z_{\min}} \frac{dz'}{z'} \int\limits_{1/(z's)} d^2 x_2 \\
&\times \left( \frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\, \theta\left(x_{10}^2 z_{\min} - x_{21}^2 z'\right) - \alpha_s(\min\{1/x_{21}^2, 1/x_{20}^2\})\, \frac{\underline{x}_{21}\cdot\underline{x}_{20}}{x_{21}^2 x_{20}^2}\, \theta\left(x_{10}^2 z_{\min} - \max\left\{x_{21}^2, x_{20}^2\right\} z'\right) \right) \\
&\times \left[ G_{21}(z', z')\, S_{20}(z') + \Gamma^{gen}_{20,21}(z', z')\, S_{21}(z') \right] \\
&+ \frac{N_c}{2\pi^2} \int\limits_{\Lambda^2/s}^{z_{\min}} \frac{dz'}{z'} \int\limits_{1/z's} d^2 x_2\, K_{\text{rcBK}}(\underline{x}_0, \underline{x}_1; \underline{x}_2)\, \theta\left(x_{10}^2 z_{\min} - x_{21}^2 z'\right) \left[ G_{21}(z', z_{\text{pol}})\, S_{20}(z') - \Gamma^{gen}_{10,21}(z', z_{\text{pol}}) \right] \\
&- \frac{N_c}{\pi^2} \int\limits_0^{z_{\text{pol}}} \frac{dz'}{z_{\text{pol}}} \int\limits_{\frac{z_{\text{pol}}}{z'(z_{\text{pol}}-z')s}} \frac{d^2 x_{32}}{x_{32}^2}\, \alpha_s\left(\frac{1}{x_{32}^2}\right) \theta(x_{10}^2 z_{\min} z_{\text{pol}} - x_{32}^2 z'(z_{\text{pol}} - z')) \\
&\qquad \times \left[ G_{\underline{x}_1 + \left(1 - \frac{z'}{z_{\text{pol}}}\right)\underline{x}_{32},\, \underline{x}_1 - \frac{z'}{z_{\text{pol}}}\underline{x}_{32}}(z_{\min}, z')\, S_{10}(z_{\min}) + \Gamma_{10,32}(z_{\min}, z') \right] \\
&+ \frac{N_c}{2\pi^2} \int\limits_0^{z_{\text{pol}}} \frac{dz'}{z_{\text{pol}}} \left( 2 - \frac{z'}{z_{\text{pol}}} + \frac{z'^2}{z_{\text{pol}}^2} \right) \int\limits_{\frac{z_{\text{pol}}}{z'(z_{\text{pol}}-z')s}} \frac{d^2 x_{32}}{x_{32}^2}\, \alpha_s\left(\frac{1}{x_{32}^2}\right) \theta(x_{10}^2 z_{\min} z_{\text{pol}} - x_{32}^2 z'(z_{\text{pol}} - z')) \\
&\times \Gamma_{10,32}(z_{\min}, z_{\text{pol}}).
\end{aligned}
$$

# Asymptotic Solution – What do we know so far?

The equations up to DLA have been analytically solved at large $N_c$ [Kovchegov et al, 2017] and numerically solved at large $N_c \& N_f$ [Kovchegov and Tawabutr, 2020].

- At large $N_c$, the quark helicity PDF has the asymptotic form

$$\Delta\Sigma(x, Q^2) \sim (1/x)^{\alpha_h^q} . \tag{9}$$

- At large $N_c \& N_f$, the asymptotic form displays oscillation pattern

$$\Delta\Sigma(x, Q^2) \sim (1/x)^{\alpha_h^q} \cos\left[\omega_q \ln(1/x) + \varphi_q\right] . \tag{10}$$

  In both cases, $\alpha_h^q \approx \frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}$, and $\omega_q$ is small and increases with $N_f$.
- Phenomenological implications were studied in [Adamiak et al, 2021].

Solutions to closed SLA equations are left for future work.

# Conclusion

- Quark's helicity contribution to proton spin follows evolution equations that contain leading DLA terms and subleading SLA terms.
- The SLA terms have been derived, with the effects of running coupling included, as the latter is also single-logarithmic.
- In the large-$N_c$ and large-$N_c \& N_f$ limits, the equations are closed but non-linear, since the unpolarized evolution is single-logarithmic.
- Future work:
  - Connection to the polarized DGLAP evolution
  - Numerical solutions to the SLA evolution equations at large $N_c$ and large $N_c \& N_f$
  - Phenomenological implications

# The End

# References


R. L. Jaffe and A. Manohar (1990)
The $g_1$ problem: Deep inelastic electron scattering and the spin of the proton
*Nucl. Phys. B* 337, 509

Y. V. Kovchegov and H. Weigert (2007)
Triumvirate of Running Couplings in Small-$x$ Evolution
*Nucl. Phys. A* 784, 188–226.

I. Balitsky (2007)
Quark contribution to the small-$x$ evolution of color dipole
*Phys. Rev. D* 75, 014001.

Y. V. Kovchegov, D. Pitonyak and M. D. Sievert (2016)
Helicity Evolution at Small-$x$
*JHEP* 1601, 072; *JHEP* 1610, 148.

Y. V. Kovchegov, D. Pitonyak and M. D. Sievert (2017)
Small-$x$ Asymptotics of the Quark Helicity Distribution: Analytic Results
*Phys. Lett.* B772, 136-140.

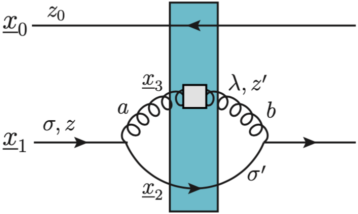
Y. V. Kovchegov and M. D. Sievert (2019)
Small-$x$ Helicity Evolution: an Operator Treatment
*Phys. Rev. D* 5, 054032.

Y. V. Kovchegov and Y. Tawabutr (2020)
Helicity at Small $x$: Oscillations Generated by Bringing Back the Quarks
*JHEP* 2008, 014.

D. Adamiak, Y. Kovchegov, W. Melnitchouk, D. Pitonyak, N. Sato and M. Sievert (2021)
First analysis of world polarized DIS data with small-$x$ helicity evolution
hep-ph/2102.06159.

# Running Coupling

- For soft unpolarized parton emission, the strong coupling constant runs the same way as the unpolarized BK evolution [Kovchegov and Weigert, 2007] [Balitsky, 2007]
- For other splitting terms, i.e. soft polarized or hard parton emission, the strong coupling constant runs with the transverse size of the daughter dipole, e.g.




$$\sim \alpha_s(1/x_{32}^2)$$

With this prescription, there is no double counting with the SLA$_T$ terms.

# Polarized Wilson Lines

$$V_{\underline{x}}^{pol} = \frac{igp_1^+}{s} \int\limits_{-\infty}^{\infty} dx^- \, V_{\underline{x}}[+\infty, x^-] \, F^{12}(x^-, \underline{x}) \, V_{\underline{x}}[x^-, -\infty]$$

$$- \frac{g^2 \, p_1^+}{s} \int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^- \, V_{\underline{x}}[+\infty, x_2^-] \, t^b \, \psi_\beta(x_2^-, \underline{x}) \, U_{\underline{x}}^{ba}[x_2^-, x_1^-] \left[\frac{1}{2} \gamma^+ \gamma^5\right]_{\alpha\beta} \bar{\psi}_\alpha(x_1^-, \underline{x}) \, t^a \, V_{\underline{x}}[x_1^-, -\infty].$$

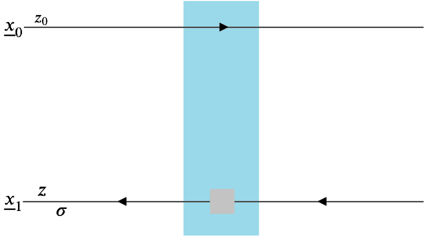
$$(U_{\underline{x}}^{pol})^{ab} = \frac{2i \, g \, p_1^+}{s} \int\limits_{-\infty}^{+\infty} dx^- \, \left(U_{\underline{x}}[+\infty, x^-] \, \mathcal{F}^{12}(x^+ = 0, x^-, \underline{x}) \, U_{\underline{x}}[x^-, -\infty]\right)^{ab}$$

$$- \frac{g^2 \, p_1^+}{s} \int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^- \, U_{\underline{x}}^{aa'}[+\infty, x_2^-] \, \bar{\psi}(x_2^-, \underline{x}) \, t^{a'} \, V_{\underline{x}}[x_2^-, x_1^-] \, \frac{1}{2} \gamma^+ \gamma_5 \, t^{b'} \, \psi(x_1^-, \underline{x}) \, U_{\underline{x}}^{b'b}[x_1^-, -\infty] + c.c..$$

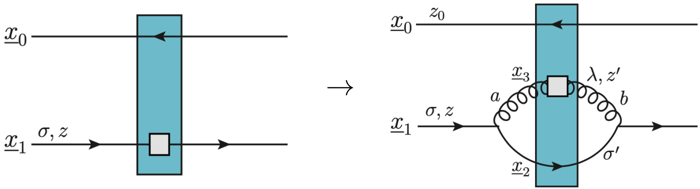
# SLA Caveats

The polarized dipole amplitudes depend on two momentum fractions.

$$\left\langle \mathrm{tr}\left[V_{\underline{x}_0} V_{\underline{x}_1}^{\dagger}(\sigma)\right]\right\rangle(z) \quad \Rightarrow \quad \left\langle \mathrm{tr}\left[V_{\underline{x}_0} V_{\underline{x}_1}^{\dagger}(\sigma)\right]\right\rangle\big(\underbrace{\min\{z, z_0\}}_{\text{minimum}}, \underbrace{z}_{\text{polarized}}\big)$$



# SLA Evolution Equations




Include not only the terms coming from the splitting of the polarized line.

- Since the evolutions of both polarized and unpolarized lines are SLA, they have to be included.
- The running coupling correction is SLA and also has to be included.

At SLA, the evolution of the dipole amplitude depends on momentum fractions of two lines – the polarized line and the softest-parton line.