Structure-dependent electromagnetic finite-size effects on the lattice

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Motivation

- Top row unitarity of CKM matrix: Study $\frac{|V_{us}|}{|V_{ud}|}$
- Leptonic decays: $P^- \to \ell^- \nu_\ell[\gamma]$ for $P = \pi, K$

$$\Gamma\left(P^- \to \ell^- \nu_\ell[\gamma]\right) = \Gamma^{\text{tree}}\left[1 + \delta R_P\right]$$

$$\Gamma^{\text{tree}} = \frac{G_F^2}{8\pi}|V_{ij}|^2 f_P^2\, m_P m_\ell^2 \left(1 - \frac{m_\ell^2}{m_P^2}\right)^2$$

- δR_P: $\alpha \neq 0$ and $m_u \neq m_d$
- Ratios of decays: $K_{\ell 2}/\pi_{\ell 2} \longrightarrow$ combine experiment and theory

$$\frac{|V_{us}|^2}{|V_{ud}|^2} = \frac{\Gamma\left(K^- \to \ell^- \nu_\ell[\gamma]\right)}{\Gamma\left(\pi^- \to \ell^- \nu_\ell[\gamma]\right)} \frac{m_{K^-}^3}{m_{\pi^-}^3} \frac{\left(m_{\pi^-}^2 - m_{\mu^-}^2\right)^2}{\left(m_{K^-}^2 - m_{\mu^-}^2\right)^2} \frac{(f_\pi/f_K)^2}{1 + \delta R_K - \delta R_\pi}$$

- Obtain theory part from lattice: reaching percent level precision
  $\Longrightarrow$ isospin breaking needed! $\Longrightarrow$ Lattice QCD+QED

QED in a finite volume

- Difficult to define charged states in finite volume with periodic boundary conditions (Gauss' law)
- Related to absence of mass gap in QED and zero-modes of photon
- We choose $\mathrm{QED_L}$: Photon zero-mode subtracted on every time slice
  [Hayakawa,Uno 2008]

$$\sum_{\mathbf{k}} \longrightarrow \sum_{\mathbf{k}}{}' = \sum_{\mathbf{k}\neq 0}$$

- Finite-volume effects: Typically larger from QED than QCD only
- Analytically: Finite-size effects in observable $\mathcal{O}(L)$ given by:

$$\Delta\mathcal{O}(L) = \mathcal{O}(L) - \mathcal{O}_{\mathrm{IV}} = \left(\frac{1}{L^3} \sum_{\mathbf{k}}{}' - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \right) \int \frac{dk_4}{2\pi} f_{\mathcal{O}}\left(k = (k_4, \mathbf{k}), \ldots\right)$$

- Soft photons travel far: Expand in small $|\mathbf{k}| = \frac{2\pi|\mathbf{n}|}{L} \implies$ expansion in L

Finite-size effects

- Massless photon $\Longrightarrow$ QED finite-size effects (FSEs):

$$\Delta\mathcal{O}(L) = \mathcal{O}(L) - \mathcal{O}_{\mathrm{IV}} = C_0 + C_{\log}\log m_P L + C_1\frac{1}{m_P L} + C_2\frac{1}{(m_P L)^2} + \dots$$

- Scaling in L is observable-dependent: e.g. self-energy $C_0 = C_{\log} = 0$
- Coefficients depend on physical particle properties: masses, charges, structure (form-factors): Point-like + structure-dependent
- What we do:
  1. FSEs in a model-independent, relativistic set-up including structure-dependence: General
  2. Derive leading structure-dependence in self-energy $(1/L^3)$ and leptonic decays $(1/L^2)$ $\longrightarrow$ Only physical quantities appear

Leptonic decays

- Infrared-divergent process:

$$\Gamma\left(P^- \to \ell^- \nu_\ell[\gamma]\right) = \Gamma_0 + \Gamma_1(\Delta E_\gamma)$$

- RM-123/Soton strategy 2015: Add and subtract point-like Γ_0^{pt}

$$\Gamma_0 + \Gamma_1(\Delta E_\gamma) = \lim_{L\to\infty}[\Gamma_0(L) - \Gamma_0^{\mathrm{pt}}(L)] + \lim_{m_\gamma\to 0}[\Gamma_0^{\mathrm{pt}}(m_\gamma) + \Gamma_1(m_\gamma, \Delta E_\gamma)]$$

- RM-123/Soton 2017: $\Gamma_0^{\mathrm{pt}}(L)$ calculated to give

$$\Gamma_0(L) - \Gamma_0^{\mathrm{pt}}(L) \sim \mathcal{O}\left(\frac{1}{L^2}\right)$$

- Our proposal: Replace $\Gamma_0^{\mathrm{pt}}(L)$ by

$$\Gamma_0^{(n)}(L) = \Gamma_0^{\mathrm{pt}}(L) + \sum_{j=2}^{n} \Delta\Gamma_0^{(j)}(L)$$

- $\Delta\Gamma_0^{(j)}(L)$ are here the FSEs of order $1/L^j$, containing both point-like and structure terms

Leptonic decays

- The residual volume-scaling is thus

$$\Gamma_0(L) - \Gamma_0^{(n)}(L) \sim \mathcal{O}\left(\frac{1}{L^{n+1}}\right)$$

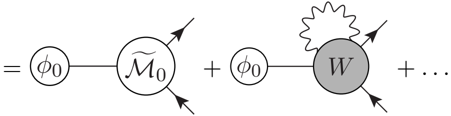
- Define the dimensionless FV function $Y^{(n)}(L)$ as

$$\Gamma_0^{(n)}(L) = \Gamma_0^{\mathrm{tree}}\left[1 + 2\frac{\alpha}{4\pi}\, Y^{(n)}(L)\right] + \mathcal{O}\left(\frac{1}{L^{n+1}}\right)$$

- NB: $Y^{(1)}(L) = Y(L)$ of [RM-123/Soton, 2017]

- Euclidean correlator for the decay $P^- \to \ell^- \nu_\ell$

$$C_W^{rs}(p, p_\ell) = \int \mathrm{d}^4 z\, e^{ipz}\, \langle \ell^-, \mathbf{p}_\ell, r; \nu_\ell, \mathbf{p}_{\nu_\ell}, s |\, \mathrm{T}[\mathcal{O}_W(z)\phi^\dagger(0)]\, |0\rangle$$

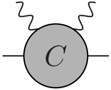


- Key to our method: Define structure-dependent kernels

The Compton scattering amplitude

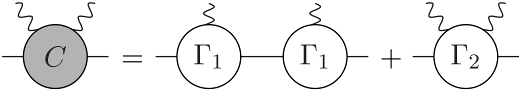
- Need to define kernels: Compton scattering amplitude

$$C_{\mu\nu}(p,k,q) =$$



$$\lim_{p^2 \to -m_{P,0}^2} C_{\mu\nu}(p,k,-k) = e^2 \int \mathrm{d}^4x \, e^{-ik\cdot x} \, \langle P, \mathbf{p} | \, T\left\{J_\mu(x) J_\nu(0)\right\} | P, \mathbf{p} \rangle$$

- Step 1: Decompose into irreducible vertex functions $\Gamma_1 = \Gamma_\mu$, $\Gamma_2 = \Gamma_{\mu\nu}$



- Amplitude $C_{\mu\nu}(p,k,q)$ satisfies Ward identities:
  - Γ_μ and $\Gamma_{\mu\nu}$ must satisfy these, but arbitrary separation!

Decomposing vertex functions

- Step 2: Form-factor decomposition (structure-dependence!)

$$\Gamma_\mu(p,k) = (2p+k)_\mu\, F(k^2,(p+k)^2,p^2) + k_\mu\, G(k^2,(p+k)^2,p^2)$$

- Contains both on-shell and off-shell dependence

$$F^{(1,0,0)}(0,-m_P^2,-m_P^2) \equiv F'(0) = -\langle r_P^2\rangle/6$$

- $F^{(0,0,1)}(0,-m_{P,0}^2,-m_{P,0}^2)$: Unphysical derivative! $\longrightarrow$ Must always cancel in the end!
- What about $G(k^2,(p+k)^2,p^2)$, $F^{(0,0,n)}(0,-m_P^2,-m_P^2)$ ...?

Decomposing vertex functions

- Step 3: Use Ward identities, e.g.

$$k_\mu \Gamma^\mu(p,k) = D(p+k)^{-1} - D(p)^{-1}$$

- Define full propagator ($Z(p^2)$: z_n [BMW 2015; RM-123/Soton 2017])

$$D(p) = \frac{Z(p^2)}{p^2 + m_P^2}$$

- Ward identity yields G as a function of F and

$$F(0, p^2, -m^2) = F(0, -m^2, p^2) = Z(p^2)^{-1}$$

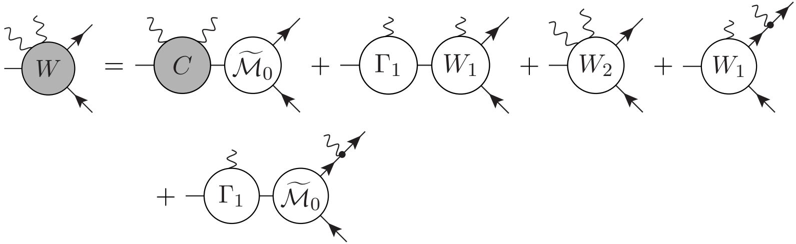
- We see e.g. $z_1 = F^{(0,0,1)}(0, -m_{P,0}^2, -m_{P,0}^2)$
- Unphysical derivative! $\longrightarrow$ Must always cancel in the end!
- Equivalently: We could put all non-physical quantities to zero directly

$$F(k^2, (p+k)^2, p^2) \to F(k^2) = 1 + k^2 F'(0) + \dots$$
$$Z(p^2) \to 1$$

- Step 4: Expand kernel functions order by order in $k \to$ arbitrary order in $1/L$

Leptonic decays

- Need to define kernels: Play the same game for $1/L^2$ effects



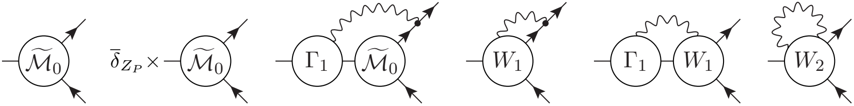
- W_1 and W_2 depend on unphysical off-shell derivatives of the decay constant: f_n [RM-123/Soton 2017]
- W_1: $A_1(k^2,(p+k)^2)$, $V_1(k^2,(p+k)^2)$: appear in $P^- \to \ell^- \nu_\ell \gamma$
- On-shell: $F_A^P = A_1(0,-m_P^2)$ and $F_V^P = V_1(0,-m_P^2)$
- Known from chiral perturbation theory [Bijnens, Ecker, Gasser 1992], lattice [RM-123/Soton 2020], experiment [...] (Discrepancies [RM-123/Soton 2020])

Diagrams

- Matrix element from reduction formula (be consistent with orders in e!)

$$\mathcal{M}^{rs} = \lim_{p^2 \to -m_P^2} Z_P^{-1} D(p)^{-1} \, C_W^{rs}(p, p_\ell)$$

- Contributions to $\mathcal{M}^{rs}$:



(a) (b)+(c) (d) (e) (f) (g)

- Use definitions of kernel functions including z_n and f_n

$$\Delta|\mathcal{M}|^2(L) = \left(\frac{1}{L^3} \sum_{\mathbf{k}}{}' - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \right) \int \frac{dk_4}{2\pi} \, f_{\mathcal{M}} \left(k = (k_4, \mathbf{k}), \mathbf{v}_\ell, \ldots \right)$$

- Expand in small $|\mathbf{k}| = \frac{2\pi|\mathbf{n}|}{L} \implies$ expansion in L

- Sum-integral differences related to finite-size coefficients $c_j(\mathbf{v}_\ell)$

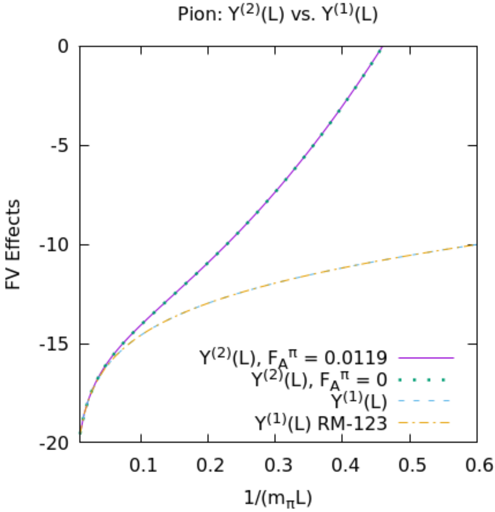
Finite-size effects

- Diagrams give $Y^{(n)}(L)$ for $n = 2$ as

$$Y^{(2)}(L) = \frac{3}{4} + 4\log\left(\frac{m_\ell}{m_W}\right) + \frac{c_3 - 2\,c_3(\mathbf{v}_\ell)}{2\pi} - 2\,A_1(\mathbf{v}_\ell) + 2\log\left(\frac{m_W L}{4\pi}\right)$$
$$- 2\,A_1(\mathbf{v}_\ell)\left[\log\left(\frac{m_P L}{4\pi}\right) + \log\left(\frac{m_\ell L}{4\pi}\right)\right] - \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\,c_2 - 4\,r_\ell^2\,c_2(\mathbf{v}_\ell)}{1-r_\ell^4}\right]$$
$$+ \frac{1}{(m_P L)^2}\left[-\frac{F_A^P}{f_P}\,\frac{4\pi\,m_P\,[(1+r_\ell^2)^2\,c_1 - 4\,r_\ell^2\,c_1(\mathbf{v}_\ell)]}{1-r_\ell^4} + \frac{8\pi\,[(1+r_\ell^2)\,c_1 - 2\,c_1(\mathbf{v}_\ell)]}{(1-r_\ell^4)}\right]$$

- All unphysical quantities vanish, i.e. we could put $f_n = z_n = 0$ from the start (as they must at all orders in $1/L$)
- Only F_A^P appears
- Charge radii $\langle r_P^2 \rangle$ cancel between diagrams due to charge conservation
- $c_j(\mathbf{v}_\ell)$ FS coefficients previously only known for $j < 3$, now for all $j \geq 3$ too

Numerical results: Physical Pion



- Perfect agreement with RM-123/Soton for $Y^{(1)}(L)$
- The $1/L^2$-correction is sizeable
- Point-like $1/L^2$ completely dominates

Conclusions

- With model-independent principles it is indeed possible to predict FSEs beyond the point-like approximation (only physical form-factors and derivatives appear)
- Self-energy $(1/L^3)$:
  - Charge radii $\langle r_P^2 \rangle$
  - Non-locality of $\mathrm{QED_L}$: Branch-cut
- Leptonic decays $(1/L^2)$:
  - Radiative leptonic decay axial form-factor F_A^P
  - Charge radii cancel because of charge conservation
- Our method is general, and new software released
  - Infrared divergent FS coefficients
- Future: Semi-leptonic decays, ...