Matrix Element Calculation

from time-ordered perturbation theory to Feynman diagrams, currents and propagators

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Motivation

The objective of this presentation is to show how to calculate matrix elements $\mathcal{M}$. This will allow us to:

- Calculate the amplitude of a process.
- Calculate the cross section of a process.
- Draw Feynman diagrams.

Matrix Elements

Let's assume we have a certain process like Compton Scattering $e^-\gamma \to e^-\gamma$. All we can observe is that an electron and a photon come in, and a scattered electron and a scattered photon come out. However, there are many ways in which this process can happen. In the leading order:

- The photon can interact with the electron, create a virtual electron which then emits a photon (**s-channel**).
- The electron can emit a photon, creating a virtual electron which then interacts with the incoming photon (**u-channel**).

Matrix Elements

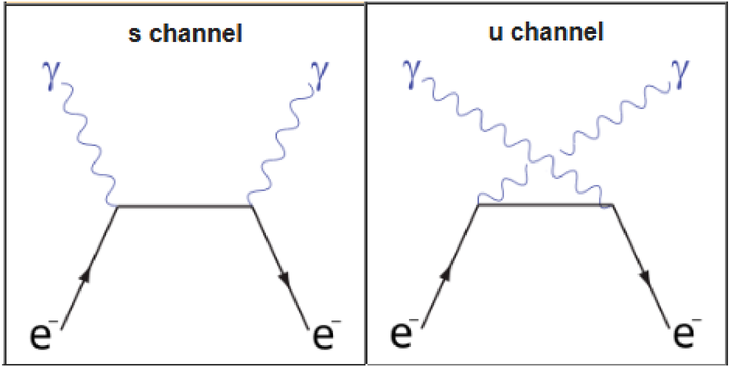


Figura: Feynman diagrams for Compton Scattering.

Fermi's Golden Rule

If we want to know the transition probability of a process, we can use Fermi's Golden Rule, which in a Lorentz invariant form is given by:

$$d\Gamma = \frac{1}{2M}|\mathcal{M}|^2\delta^4(p_i - p_f)\prod_{i=1}^{N}\frac{d^3p_i}{(2\pi)^3 2E_i}^{1} \tag{1}$$

The challenge then is to calculate to sum the individual contributions to the matrix element $\mathcal{M}$. In order to do that, we can use the relations:

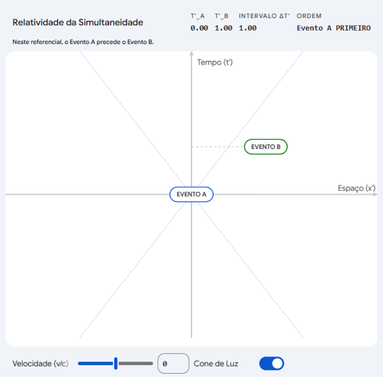
$$\mathcal{M}_{fi}^{ab} = \langle\psi_a|\hat{H}'|\psi_b\rangle = (4E_aE_b)^{1/2}\,T_{fi}^{ab} \tag{2}$$

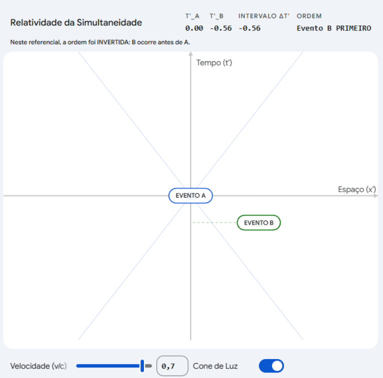
$$T_{fi}^{ab} = \frac{\langle f|H'|j\rangle\langle j|H'|i\rangle}{E_i - E_j} \tag{3}$$

[1] The factor of $1/2M$ is only true for a single particle in the initial state

Time ordering issue

There is however a problem with the previous equations, they imply an "objective" order for the interactions, when in reality, different observers may see the interactions in different orders and therefore reach different conclusions about the process.
This violates the rules of special relativity, since it implies that there is a preferred frame of reference.





Feynman propagators

If we sum both contributions to the matrix element for the Compton scattering process, we get:

$$T_{fi} = \frac{\langle f|H'|j\rangle\langle j|H'|i\rangle}{E_i - E_j} + \frac{\langle f|H'|j'\rangle\langle j'|H'|i\rangle}{E_i - E_{j'}} = \tag{4}$$

$$= \frac{\langle e_f^- \gamma_f|H'|e_{v1}^-\rangle\langle e_{v1}^-|H'|e_i^- \gamma_i\rangle}{(E_{e_i} + E_{\gamma_i}) - E_{e_{v1}}} + \frac{\langle e_f^- \gamma_f|H'|e_{v2}^- \gamma_i \gamma_f\rangle\langle e_{v2}^- \gamma_i \gamma_f|H'|e_i^- \gamma_i\rangle}{(E_{e_i} + E_{\gamma_i}) - (E_{e_{v2}} + E_{\gamma_i} + E_{\gamma_f})} \tag{5}$$

$$\mathcal{M}_{fi}^{ab} = \frac{1}{2E_X}\frac{g_a g_b}{E_a - E_c - E_X} \quad \mathcal{M}_{fi}^{ba} = \frac{1}{2E_X}\frac{g_a g_b}{E_b - E_d + E_X} \tag{6}$$

$$\mathcal{M}_{fi} = \frac{g_a g_b}{(E_a - E_c)^2 - E_X^2} = \frac{g_a g_b}{q^2 - m_X^2} \tag{7}$$

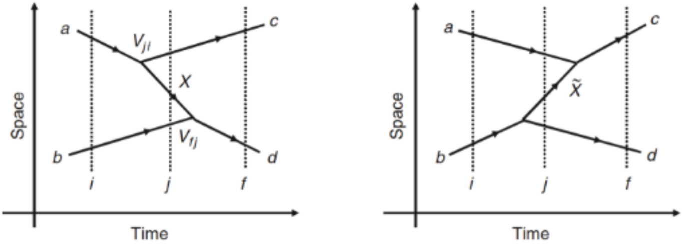


Figura: Simple example used in the slides.

Currents

Although the previous example was simple, it cannot be applied to something like the Compton scattering process, since the particles aren't scalars, but rather spinors and vectors. In order to solve this problem, we can use the concept of currents, which are defined as:

$$J^{\mu}_{fi} = -e\bar{u}(p_f)\gamma^{\mu}u(p_i) \tag{8}$$

Where $\bar{u}(p_f)$ and $u(p_i)$ are the spinors of the final and initial electrons, respectively, and γ^{μ} are the Dirac matrices.These currents in practice replace the "g_s". The propagator is also changed to:

$$\frac{\gamma^{\mu}q_{\mu} + m}{q^2 - m^2} \tag{9}$$

The final result

This process only needs three pieces of information to be calculated:

- The currents, which depend on the particles involved in the process.
- The propagators, which depend on the virtual particles involved in the process.
- The coupling constants, which depend on the interactions involved in the process.

This makes it a very powerful tool, since it allows us to calculate the amplitude/cross section of any process, as long as we have these pieces.