Fermi's Golden Rule

Pedro Gralhós

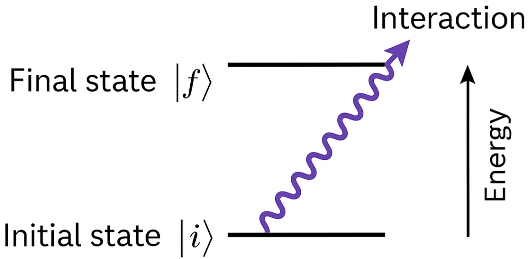
University of Coimbra

Introduction

- Classical physics: trajectories
- Quantum physics: transition probabilities

What do experiments actually measure in particle physics?

Quantum Transitions



Quantum Transitions

We start in an initial state $|i\rangle$, and some *interaction* triggers a transition to the final state $|f\rangle$.
What we want to know is the **probability of this transition occurring due to a specific interaction**.

Fermi's Golden Rule

$$\Gamma_{i\to f} \;=\; 2\pi\,|T_{fi}|^2\,\rho(E_f)$$

- $\Gamma_{i\to f}$: transition rate
- T_{fi}: matrix element
- $\rho(E_f)$: density of final states

The Transition Matrix Element

$$T_{fi} = \langle f | H' | i \rangle$$

- First–order matrix element of the interaction
- Connects the initial state $|i\rangle$ to the final state $|f\rangle$

Density of Final States

$$\rho(E_f) = \frac{dn}{dE}$$

- Number of final states per unit energy
- Depends only on kinematics
- Not on the interaction

Therefore, we can say that the transition rate is:

$$\textit{Transition rate} \propto \textit{Dynamics} \times \textit{Kinematics}$$

Counting Final States

Assuming that the particles are described by plane wave functions

$$\psi_{\mathbf{p}}(\mathbf{x}) = \frac{1}{\sqrt{V}}\, e^{i\mathbf{p}\cdot\mathbf{x}},$$

the wavefunctions can be normalized to 1 particle per box volume of side a.

This implies periodic boundary conditions:

$$\psi_{\mathbf{p}}(\mathbf{x}) = \psi_{\mathbf{p}}(\mathbf{x}+\mathbf{a}).$$

Counting Final States

Thus, the momentum $\mathbf{p}$ must be quantized in all directions:

$$\mathbf{p} = \left(\frac{2\pi n_x}{a}, \frac{2\pi n_y}{a}, \frac{2\pi n_z}{a} \right). \qquad n_x, n_y, n_z \in \mathbb{Z}$$

So each momentum state occupies a volume $d^3\vec{p}$ in momentum space

$$d^3\vec{p} = dp_x\, dp_y\, dp_z = \left(\frac{2\pi}{a} \right)^3 = \left(\frac{2\pi}{V} \right)^3.$$

This allows us to relate n and $\mathbf{p}$:

$$dn = \frac{d^3\vec{p}}{(2\pi)^3} V.$$

But this is not Lorentz invariant!

Lorentz Invariant Phase Space

Length varies by a factor of $\frac{1}{\gamma}$ in the direction of motion. The solution is to **normalize the wave function to $2E$ particles per unit volume**.

A new Lorentz invariant matrix element, $\mathcal{M}_{fi}$ can then be defined. Consider a general process $a + b \to 1 + 2$. The matrix element $\mathcal{M}$ takes the form:

$$\mathcal{M}_{fi} = \sqrt{2E_1 \cdot 2E_2 \cdot 2E_a \cdot 2E_b}\; T_{fi}$$

Lorentz Invariant Phase Space

For each final-state particle:

$$\frac{d^3\mathbf{p}}{(2\pi)^3\,2E}$$

For a general final state:

$$d\Phi = \prod_f \frac{d^3\mathbf{p}_f}{(2\pi)^3\,2E_f}$$

Lorentz Invariant Phase Space

The factors of the form $\frac{d^3\vec{p}}{(2\pi)^3 2E}$ are are called **Lorentz-invariant phase space factor**.

Invariant Form of Fermi's Golden Rule

Generalizing to a process $a \to 1 + 2 + \ \dots \ + n$, the final form is:

$$\Gamma_{i\to f} = \frac{(2\pi)^4}{2E_a} \int |M_{if}|^2 \delta(E_a - \sum_i^n E_i).\delta^3(\vec{p}_a - \sum_i^n \vec{p}_i) \prod_i^n \frac{d^3\vec{p}_i}{(2\pi)^3 2E_i}$$

- The Dirac deltas enforce conservation laws

From transition rate to lifetime

The transition rate Γ has a direct physical interpretation. It determines how fast a particle decays.
Two important physical quantities can be defined:

- **Mean lifetime** $\tau = 1/\Gamma$
  - Average time a particle exists before decaying
- **Half-life** $T_{1/2} = \ln 2/\Gamma$
  - Time after which half of an initial population has decayed

These quantities are directly measurable in laboratory experiments.