In-medium propagation of particles in an Open Quantum System (OQS) approach

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Summer 2025

Motivation

- The Quark-Gluon Plasma (QGP) is a state of matter where quarks and gluons are in an extreme condition such that they are deconfined
- The state of the art formalism which describes propagation of jet particles in the QGP is the BDMPS-Z. A full analytical solution is still restricted to the eikonal approximation
- The BMDPS-Z model escalates in complexity when one tries to lift approximations, yielding challenging equations even in a numerical approach
- Since we can think of the jet and the QGP as two systems interacting, one can think of a Open Quantum System formulation as a candidate for a more natural framework
- In our work, we will introduce the main ideas of OQS and discuss some simple models inspired by the interaction of jets with a QGP

Density matrix formalism

Allows for a description of statistical mixtures of quantum states. A quantum state is represented by an operator $\rho : \mathcal{H} \to \mathcal{H}$ satisfying:

$$(1) \quad \rho^\dagger = \rho \qquad (2) \quad \mathrm{tr}(\rho) = 1 \qquad (3) \quad \langle\psi|\rho|\psi\rangle \geq 0 \;\; \forall\, \psi \in \mathcal{H}$$

Expected value of the observable A when the quantum state is ρ: $\langle A\rangle = \mathrm{tr}(A\rho)$

Pure and Mixed States are identified with:

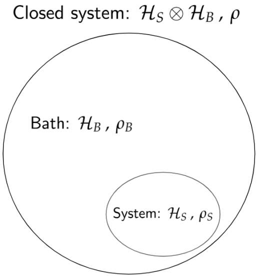
Pure states: $\rho = |\psi\rangle\langle\psi|$ Mixed states: $\rho = \sum_i p_i \left|\psi_i\middle\rangle\middle\langle\psi_i\right|$

Liouville-von Neumann Equation

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H(t), \rho(t)] \tag{1}$$

Open Quantum Systems: Markovian evolution

Quantum system interacting with a much larger bath.
Goal: find an effective equation for the time evolution of $\rho_S = \mathrm{tr}_B(\rho)$.



For a Markovian time evolution, this equation has a specific form [1]:

Lindblad equation

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H(t), \rho(t)] + \sum_k \gamma_k(t) \left[V_k(t)\rho(t)V_k^\dagger(t) - \frac{1}{2}\{V_k^\dagger(t)V_k(t), \rho(t)\} \right] \quad (2)$$

V_k are called the jump operators.

Open Quantum Systems: Weak coupling limit

We can obtain the Lindblad equation in the regime where the system-bath interaction is weak [1].
Total Hamiltonian: $H = H_S + H_B + V$. Assume that $V = \sum_k A_k \otimes B_k$, $A_k^\dagger = A_k$ and $B_k^\dagger = B_k$

Lindblad equation

$$\frac{d\rho_S(t)}{dt} = -\frac{i}{\hbar}[H_S + H_{LS}, \rho_S(t)] + \sum_{\omega,k,\ell} \gamma_{k\ell}(\omega) \left[A_\ell(\omega)\rho_S(t)A_k^\dagger(\omega) - \frac{1}{2}\{A_k^\dagger(\omega)A_\ell(\omega), \rho_S(t)\}\right]$$

where the jump operators are given by:

$$A_k(\omega) = \sum_{\epsilon'-\epsilon=\omega} \left|\psi_\epsilon\right\rangle\!\left\langle\psi_\epsilon\right| A_k \left|\psi_{\epsilon'}\right\rangle\!\left\langle\psi_{\epsilon'}\right| \tag{3}$$

where $\left|\psi_\epsilon\right\rangle$ are the eigenstates of H_S with energy ϵ. The other quantities are given by::

$$\gamma_{kl} = 2\pi \mathrm{tr}(B_k(\omega)B_l\rho_{th}) \quad H_{LS} = \sum_{\omega,k,\ell} S_{k\ell}(\omega)A_k^\dagger(\omega)A_\ell(\omega) \quad S_{kl}(\omega) = \mathrm{P.V.}\int_{-a}^{a} \frac{\mathrm{Tr}\left[B_k(\omega')B_l\rho_B\right]}{(\omega-\omega')}\, d\omega' \tag{4}$$

where $B_k(\omega)$ is the Fourier transform of B_k in the interaction picture.

Phenomenological Models of Splitting

We represent the 1-particle state by $|1\rangle$ and the 2-particle state by $|2\rangle$.

- Hilbert space: $\mathcal{H} = \text{span}(\{|1\rangle, |2\rangle\}) \implies \rho(t) = \begin{pmatrix} a(t) & b(t) \\ b(t)^* & 1 - a(t) \end{pmatrix}$
- Energy conservation $\implies H_S = \epsilon \mathbb{1} \implies [H_S, \rho] = 0$

We code the transition $|1\rangle \to |2\rangle$ in the Jump operator $L = |2\rangle\langle 1| \implies$

Lindblad Equation

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}\cancel{[H_S, \rho]} + \gamma \left(\begin{pmatrix} 0 & 0 \\ 0 & a(t) \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 2a(t) & b(t) \\ b(t) & 0 \end{pmatrix} \right) \tag{5}$$

$$\rho(0) = |1\rangle\langle 1| \implies \rho(t) = \begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & 1 - e^{-\gamma t} \end{pmatrix} \tag{6}$$

Spin and Colour Dynamics in Parton Splitting

An easy generalization of the previous result allows partons to have:

Spin: $\mathcal{H} = \text{span}(\{|1,\uparrow\rangle, |1,\downarrow\rangle, |2,\uparrow\rangle, |2,\downarrow\rangle\})$

$$\rho(t) = \left(\begin{array}{cc|cc} e^{-\gamma_\uparrow t}a(0) & e^{-\frac{\gamma_\uparrow+\gamma_\downarrow}{2}}b(0) & \multicolumn{2}{c}{\multirow{2}{*}{\mathbf{0}}} \\ e^{-\frac{\gamma_\uparrow+\gamma_\downarrow}{2}t}b(0)^* & e^{-\gamma_\downarrow t}(1-a(0)) & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{\mathbf{0}}} & (1-e^{-\gamma_\uparrow t})a(0) & 0 \\ & & 0 & (1-e^{-\gamma_\downarrow t})(1-a(0)) \end{array}\right) \tag{7}$$

Color: $\mathcal{H} = \text{span}(\{|1,r\rangle, |1,g\rangle, |1,b\rangle, |2,r\rangle, |2,g\rangle, |2,b\rangle\})$

$$\rho(t) = \left(\begin{array}{ccc|ccc} e^{-\gamma_r t}a(0) & e^{-\frac{\gamma_r+\gamma_g}{2}}b(0) & e^{-\frac{\gamma_r+\gamma_b}{2}}c(0) & & & \\ e^{-\frac{\gamma_r+\gamma_g}{2}t}b(0)^* & e^{-\gamma_g t}d(0) & e^{-\frac{\gamma_g+\gamma_b}{2}t}f(0) & & \mathbf{0} & \\ e^{-\frac{\gamma_r+\gamma_b}{2}}c(0)^* & e^{-\frac{\gamma_g+\gamma_b}{2}t}f(0)^* & e^{-\gamma_b t}(1-a(0)-d(0)) & & & \\ \hline & & & (1-e^{-\gamma_r t})a(0) & 0 & 0 \\ & \mathbf{0} & & 0 & (1-e^{-\gamma_g t})d(0) & 0 \\ & & & 0 & 0 & (1-e^{-\gamma_b t})(1-a(0)-d(0)) \end{array}\right) \tag{8}$$

Splitting in the Position and Momentum Spaces

We consider the restricted Fock space for distinguishable particles:

$$\mathcal{H} = \mathcal{H}_1 \oplus (\mathcal{H}_2 \otimes \mathcal{H}_3), \quad \mathcal{H}_i = L^2(\mathbb{R}^3)$$

For each $\vec{k'} \in \mathbb{R}^3$, we consider the Jump operator $L_{\vec{k'}}$ which acts on the states $|\mathbf{\Psi}_{1,\vec{k}_1}\rangle = (e^{i\vec{k}_1 \cdot \vec{x}_1}, 0)$ and $|\mathbf{\Psi}_{2,\vec{k}_2;\vec{k}_3}\rangle = (0, e^{i\vec{k}_2 \cdot \vec{x}_2} e^{i\vec{k}_3 \cdot \vec{x}_3})$ in the following way:

$$L_{\vec{k'}} \left| \mathbf{\Psi}_{1,\vec{k}_1} \right\rangle = \left| \mathbf{\Psi}_{2,\vec{k'};\vec{k}_1 - \vec{k'}} \right\rangle \quad (1) \qquad L_{\vec{k'}} \left| \mathbf{\Psi}_{2,\vec{k}_2;\vec{k}_3} \right\rangle = (0,0) \quad (2)$$

System Hamiltonian

$$H_S = -\frac{\hbar^2}{2m_1} \nabla^2_{\vec{x}_1} \psi_1 \oplus \left(-\frac{\hbar^2}{2m_2} \nabla^2_{\vec{x}_2} \psi_2 \otimes \mathbb{1}_3 + \mathbb{1}_2 \otimes -\frac{\hbar^2}{2m_3} \nabla^2_{\vec{x}_3} \psi_3 \right) \qquad (9)$$

Splitting in the Position and Momentum Spaces

Lindblad equation

$$\frac{\partial\rho}{\partial t} = -\frac{i}{\hbar}[H_S,\rho] + \int_{\mathbb{R}^3}\gamma(\vec{k}')\left(L_{\vec{k}'}\rho L_{\vec{k}'}^\dagger - \frac{1}{2}\{L_{\vec{k}'}^\dagger L_{\vec{k}'},\rho\}\right)d^3\vec{k}' \tag{10}$$

Probability density of finding the system in a 1-particle state with a given momentum: $\rho_{1,\vec{k}_1,1,\vec{k}_1}(t) \equiv \langle \mathbf{\Psi}_{1,\vec{k}_1}|\rho|\mathbf{\Psi}_{1,\vec{k}_1}\rangle$. Probability density of finding the system in a 2-particle state with a given momentum for each particle: $\langle \mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}|\rho|\mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}\rangle$.

$$\dot{\rho}_{1,\vec{k}_1,1,\vec{k}_1}(t) = -\left(\int_{\mathbb{R}^3}\gamma(\vec{k}')d^3\vec{k}'\right)\rho_{1,\vec{k}_1,1,\vec{k}_1}(t) \tag{11}$$

$$\left\langle \mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}\middle|\,\dot{\rho}(t)\,\middle|\mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3}\right\rangle = \gamma(\vec{k}_2)\,\rho_{1,\vec{k}_2+\vec{k}_3,1,\vec{k}_2+\vec{k}_3}(t) \tag{12}$$

Splitting in the Position and Momentum Spaces

We define $\Gamma \equiv \int_{\mathbb{R}^3} \gamma(\vec{k}')d^3\vec{k}'$. By explicitly solving the equations, we obtain:

$$\left\langle \mathbf{\Psi}_{1,\vec{k}_1} \middle| \rho(t) \middle| \mathbf{\Psi}_{1,\vec{k}_1} \right\rangle = e^{-\Gamma t} \left\langle \mathbf{\Psi}_{1,\vec{k}_1} \middle| \rho(0) \middle| \mathbf{\Psi}_{1,\vec{k}_1} \right\rangle \tag{13}$$

$$\left\langle \mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3} \middle| \rho(t) \middle| \mathbf{\Psi}_{2,\vec{k}_2,\vec{k}_3} \right\rangle = \frac{\gamma(\vec{k}_2)}{\Gamma}(c - e^{-\Gamma t}) \left\langle \mathbf{\Psi}_{1,\vec{k}_2+\vec{k}_3} \middle| \rho(0) \middle| \mathbf{\Psi}_{1,\vec{k}_2+\vec{k}_3} \right\rangle \tag{14}$$

Initial condition: $\rho(0) = |\mathbf{\Psi}_0\rangle \langle \mathbf{\Psi}_0|$, with $|\mathbf{\Psi}_0\rangle = (\psi_1(\vec{x}_1), 0) \in \mathcal{H}$ (a pure state of a single particle) $\implies c = 1$.

Splitting as a result of system-bath interaction

Goal: derive a model for the bath and its interaction with the system that reproduces the previous results.

Following [1], we consider a bath which is a continuum of harmonic oscillators:

$$H = H_S + H_B + V = \epsilon \mathbb{1}_S \otimes \mathbb{1}_B + \mathbb{1}_S \otimes \hbar \int_0^\infty a^\dagger(\omega) a(\omega) D(\omega) d\omega + V \qquad (15)$$

We need to find A_k and B_k such that $V = \sum_k A_k \otimes B_k$, $A_k^\dagger = A_k$ and $B_k^\dagger = B_k$

Jump operators in the degenerate case: $A_k(\omega = 0) = \sum_{n,m=1,2} |n\rangle\langle n| A_k |m\rangle\langle m| = A_k$

System should be degenerate $\implies H_{LS} \propto \mathbb{1}$

One possible choice:

$$A_1 = \sigma_x \qquad B_1 = \int_{-\omega_{\max}}^{\omega_{\max}} B(\omega)\, d\omega, \text{with} \begin{cases} B(\omega) & = h(\omega) a_\omega, \\ B(-\omega) & = h(\omega) a_\omega^\dagger, \end{cases} \text{for } \omega > 0 \qquad (16)$$

$$\implies V = \sigma_x \otimes \int_0^{\omega_{\max}} h(\omega)(a_\omega^\dagger + a_\omega) d\omega = \int_0^{\omega_{\max}} h(\omega)(L + L^\dagger)(a_\omega^\dagger + a_\omega) d\omega \qquad (17)$$

Splitting as a result of system-bath interaction

$$\gamma_{11}(\omega) = 2\pi \text{tr}(B_1(\omega) B_1 \rho_{th}) = \begin{cases} 2\pi h^2(\omega_0)[\bar{n}(\omega_0)+1], & \omega > 0 \\ 2\pi h^2(\omega_0)\bar{n}(\omega_0), & \omega < 0 \end{cases} \tag{18}$$

Considering a high temperature regime, $\bar{n}(\omega)+1 \approx \bar{n}(\omega)$ and an Ohmic spectral density, $J(\omega) = h^2(\omega) = \eta\omega\theta(\omega_{max} - \omega)$, we calculate $\gamma_{11}(0)$ as the $\omega \to 0$ limit of this expression:

$$\begin{aligned} \gamma_{11} &= \lim_{\omega_0 \to 0} 2\pi J(\omega_0)\bar{n}(\omega_0) \\ &= 2\pi \lim_{\omega_0 \to 0} \frac{\eta\omega_0\theta(\omega_{max} - \omega_0)}{e^{\omega_0/T} - 1} = 2\pi\eta T \end{aligned} \tag{19}$$

Splitting as a result of system-bath interaction

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}\cancel{[\sigma_x\sigma_x S_{11}(0), \rho(t)]} + \gamma_{11}(0)\left(\sigma_x\rho(t)\sigma_x - \tfrac{1}{2}\{\sigma_x\sigma_x, \rho(t)\}\right)$$

$$\begin{pmatrix} \frac{\partial a}{\partial t}(t) & \frac{\partial b}{\partial t}(t) \\ \frac{\partial b^*}{\partial t}(t) & -\frac{\partial a}{\partial t}(t) \end{pmatrix} = \gamma_{11}(0)\begin{pmatrix} d(t) - a(t) & -i\,\mathrm{Im}[b(t)] \\ i\,\mathrm{Im}[b(t)] & a(t) - d(t) \end{pmatrix}, \quad \gamma_{11}(0) = 2\pi\eta T \tag{20}$$

$$\rho(t) = \begin{pmatrix} \frac{1}{2}\left((a_0 + d_0) + (a_0 + d_0)e^{-2\gamma t}\right) & x_0 + y_0 e^{-2\gamma t} \\ x_0 - y_0 e^{-2\gamma t} & \frac{1}{2}\left((a_0 + d_0) - (a_0 + d_0)e^{-2\gamma t}\right) \end{pmatrix} \tag{21}$$

Splitting as a result of system-bath interaction

If we impose that the system starts in the pure state $|1\rangle\langle 1|$, we obtain:

$$\rho(t) = \begin{pmatrix} \frac{1}{2}\left(1 + e^{-2\gamma t}\right) & 0 \\ 0 & \frac{1}{2}\left(1 - e^{-2\gamma t}\right) \end{pmatrix} \tag{22}$$

$$\lim_{t \to \infty} \rho(t) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{23}$$

Problem: the system evolves towards the maximally mixed state, with maximum entropy. This is because we considered a reversible splitting process, whereas splitting is known to be irreversible. In the interaction Hamiltonian there are $|1\rangle \to |2\rangle$ and $|2\rangle \to |1\rangle$ transitions. The latter would correspond to two particles fusing into one, a process we do not consider in our phenomenological splitting model.

Imposing irreversibility

A possible solution is to consider an effective non-hermitian Hamiltonian:

$$V = \int_0^{\omega_{\max}} h(\omega)\sigma_-(a_\omega^\dagger + a_\omega)d\omega \tag{24}$$

We now consider $A_1 = \sigma_- = L$ (in the $\{|1\rangle, |2\rangle\}$ basis) and keep B_1 the same $\Rightarrow$ $\gamma_{11}(0) = 2\pi\eta T$ as before.
We obtain a Lindblad equation similar to 5, with an extra term due to H_{LS}:

$$\begin{aligned}\frac{d}{dt}\rho_A(t) &= -\frac{i}{\hbar}\left[H + H_{LS}, \rho(t)\right] + \sum_{k,\ell}^{1} \gamma_{k\ell}(0)\left(A_\ell(0)\,\rho_A(t)\,A_k^\dagger(0) - \tfrac{1}{2}\left\{A_k^\dagger(0)A_\ell(0), \rho_A(t)\right\}\right) \\ &= -\frac{i}{\hbar}[S_{11}(0)L^\dagger L, \rho_A(t)] + \gamma_{11}(0)\left(L\rho_A(t)L^\dagger - \tfrac{1}{2}\{L^\dagger L, \rho_A(t)\}\right)\end{aligned} \tag{25}$$

This equation describes an irreversible process as we desired, but introduces a Lamb-shift of $S_{11}(0)$ between the 2 levels.

Imposing Irreversibility

Considering an Ohmic spectral density, we can calculate this shift:

$$S_{11}(0) = -\text{P.V.} \int_{-\omega_{\text{max}}}^{\omega_{\text{max}}} d\omega' \frac{\text{Tr}\left[B_1(\omega')B_1\rho_B\right]}{\omega'} = -\frac{\omega_{max}}{4} \tag{26}$$

The Lindblad equation reduces to a form similar to equation 5, but with this new term:

$$\frac{\partial\rho}{\partial t} \equiv \begin{pmatrix} \frac{\partial a}{\partial t}(t) & \frac{\partial b}{\partial t}(t) \\ \frac{\partial b^*}{\partial t}(t) & -\frac{\partial a}{\partial t}(t) \end{pmatrix} = \frac{i}{\hbar} \begin{pmatrix} 0 & -S_{11}(0)b(t) \\ S_{11}(0)b(t)^* & 0 \end{pmatrix} + \gamma \begin{pmatrix} -a(t) & -\frac{b(t)}{2} \\ -\frac{b(t)^*}{2} & a(t) \end{pmatrix} \tag{27}$$

Imposing the initial condition $\rho(0) = |1\rangle\langle 1|$ we obtain the same result as before:

$$\rho(t) = \begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & 1 - e^{-\gamma t} \end{pmatrix}. \tag{28}$$

Next steps

- The non-hermitian Hamiltonian seems like an *ad hoc* solution that also introduces problems in the form of the Lamb-shift. There should be a more fundamental reason for the irreversible behavior.
- Describe the system-bath interaction in the continuous position and momentum basis.
- Quantify the coupling in the vacuum and QGP to calculate the difference in decay rates between the two cases.

References

[1] Ángel Rivas and Susana F. Huelga. Open Quantum Systems: An Introduction. Springer Berlin Heidelberg, 2012. ISBN: 9783642233548. DOI: 10.1007/978-3-642-23354-8. URL: http://dx.doi.org/10.1007/978-3-642-23354-8.