Casimir's Trace Tricks to calculate $\langle |M_{fi}|^2 \rangle$

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The electron-positron annihilation

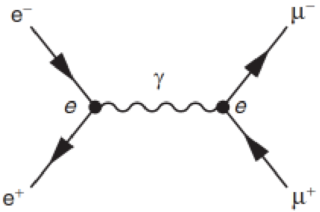

Figure: Lowest Order diagram[1] for the QED - $e^+ + e^- \rightarrow \mu^+ + \mu^-$

- $e^+ + e^- \rightarrow \mu^+ + \mu^-$;
- From the Diagram:

$$M_{fi} = [\bar{\nu}(p_2)(ie\gamma^\mu)u(p_1)]\,\frac{g_{\mu\nu}}{q^2}\,[\bar{u}(p_3)(ie\gamma^\nu)\nu(p_4)]$$

$$= -\frac{e^2}{q^2}\,[\bar{\nu}(p_2)\gamma^\mu u(p_1)]\,[\bar{u}(p_3)\gamma_\mu \nu(p_4)]$$

[1]M. Thomson, Modern Particle Physics, Section 6.2

The electron-positron annihilation

$$|M_{fi}|^2 = \frac{e^4}{q^4}\left[\bar{\nu}(p_2)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_3)\gamma_\mu \nu(p_4)\right]$$
$$\left[\bar{\nu}(p_2)\gamma^\nu u(p_1)\right]^\dagger \left[\bar{u}(p_3)\gamma_\nu \nu(p_4)\right]^\dagger$$

- For two spinors, ψ and ϕ we have that[2]:

$$\left[\bar{\psi}\gamma^\mu \phi\right]^\dagger = \left[\psi^\dagger \gamma^0 \gamma^\mu \phi\right]^\dagger = \phi^\dagger \gamma^{\mu\dagger}\gamma^{0\dagger}\psi = \phi^\dagger \gamma^0\gamma^0\gamma^{\mu\dagger}\gamma^0 \psi = \bar{\phi}\gamma^\mu \psi$$

$$\Rightarrow |M_{fi}|^2 = \frac{e^4}{q^4}\left[\bar{\nu}(p_2)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_1)\gamma^\nu \nu(p_2)\right]$$
$$\left[\bar{u}(p_3)\gamma_\mu \nu(p_4)\right]\left[\bar{\nu}(p_4)\gamma_\nu\, u(p_3)\right]$$

[2] Note that $\gamma^0\gamma^0 = \mathbf{1}$ and $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$

Spin Sums - Unpolarized Case

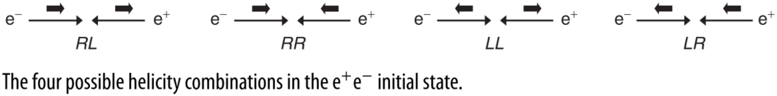

Figure: Possible helicity combinations for the initial state[3]

- The final state of $\mu^+\mu^-$ also has 4 possible helicity states. Thus, there are 16 possible orthogonal helicity states.

[3]M. Thomson, Modern Particle Physics, Section 6.2.1

Spin Sums - Unpolarized Case[4]

- No polarization $\rightarrow$ All initial states are equally probable $\rightarrow \frac{1}{4}$

$$\begin{aligned}\langle|M_{fi}|^2\rangle &= \frac{1}{4}\left(|M_{RR}|^2+|M_{RL}|^2+|M_{LR}|^2+|M_{LL}|^2\right)\\ &= \frac{1}{4}\left(|M_{RR\to RR}|^2+|M_{RR\to RL}|^2+\cdots+|M_{RL\to RR}|^2+\cdots\right)\\ &\Rightarrow \langle|M_{fi}|^2\rangle = \frac{1}{4}\sum_{spins}|M|^2\end{aligned}$$

$$\begin{aligned}\langle|M_{fi}|^2\rangle = \frac{e^4}{4q^4}&\sum_{s,r}\left[\bar{\nu}^r(p_2)\gamma^\mu u^s(p_1)\right]\left[\bar{u}^s(p_1)\gamma^\nu \nu^r(p_2)\right]\\ &\sum_{s',r'}\left[\bar{u}^{s'}(p_3)\gamma_\mu \nu^{r'}(p_4)\right]\left[\bar{\nu}^{s'}(p_4)\gamma_\nu\, u^{r'}(p_3)\right]\\ &= \frac{e^4}{4q^4}\mathcal{L}^{\mu\nu}_{(e)}\mathcal{L}^{(\mu)}_{\mu\nu}\end{aligned}$$

[4] M. Thomson, Modern Particle Physics, Section 6.2.1 and 6.5.2

Spin Sums and Trace Formalism[5]

$$\Rightarrow \mathcal{L}_{(e)}^{\mu\nu} = \sum_{s,r} \left[\bar{\nu}^r(p_2)\gamma^\mu u^s(p_1)\right]\left[\bar{u}^s(p_1)\gamma^\nu \nu^r(p_2)\right]$$

$$= \sum_{s,r=1}^{2} \left[\bar{\nu}_j^r(p_2)\gamma_{ji}^\mu u_i^s(p_1)\right]\left[\bar{u}_n^s(p_1)\gamma_{nm}^\nu \nu_m^r(p_2)\right]$$

$$= \left[\sum_{r=1}^{2} \nu_m^r(p_2)\bar{\nu}_j^r(p_2)\right]\left[\sum_{s=1}^{2} u_i^s(p_1)\bar{u}_n^s(p_1)\right]\gamma_{ji}^\mu \gamma_{nm}^\nu$$

[5] M. Thomson, Modern Particle Physics, Section 6.5.2

Trace Formalism and Completeness Relations

The Completeness Relations are as follows[6]:

- $\sum_{s=1}^{2} u_s \bar{u}_s = (\gamma\,\mu\, p_\mu + mI) = \mathbf{p} + m;$
- $\sum_{r=1}^{2} \nu_r \bar{\nu}_r = (\gamma\,\mu\, p_\mu - mI) = \mathbf{p} - m.$

Replacing these in the previous expression:

$$\begin{aligned}
\Leftrightarrow \mathcal{L}_{(e)}^{\mu\nu} &= \left[\sum_{r=1}^{2} \nu_m^r(p_2)\bar{\nu}_j^r(p_2)\right] \gamma_{ji}^{\mu} \left[\sum_{s=1}^{2} u_i^s(p_1)\bar{u}_n^s(p_1)\right] \gamma_{nm}^{\nu} \\
&= (\mathbf{p_2} - m)_{mj}\gamma_{ji}^{\mu}(\mathbf{p_1} + m)_{in}\gamma_{nm}^{\nu} \\
&= [(\mathbf{p_2} - m)\gamma^{\mu}(\mathbf{p_1} + m)\gamma^{\nu}]_{mm} \\
&= Tr\left[(\mathbf{p_2} - m)\gamma^{\mu}(\mathbf{p_1} + m)\gamma^{\nu}\right]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \mathcal{L}_{\mu\nu}^{(\mu)} &= \sum_{s',r'} \left[\bar{u}^{s'}(p_3)\gamma_\mu \nu^{r'}(p_4)\right] \left[\bar{\nu}^{s'}(p_4)\gamma_\nu\, u^{r'}(p_3)\right] \\
&= Tr\left[(\mathbf{p_3} + M)\gamma_\mu(\mathbf{p_4} - M)\gamma_\nu\right]
\end{aligned}$$

[6] M. Thomson, Modern Particle Physics, Section 6.5.1

Spin Sums and Trace Formalism

Putting everything together we get:

$$\langle |M_{fi}|^2 \rangle = \frac{e^4}{4q^4} \mathcal{L}^{\mu\nu}_{(e)} \mathcal{L}^{(\mu)}_{\mu\nu}$$
$$= \frac{e^4}{4q^4} Tr\left[(\mathbf{p_2} - m)\gamma^\mu(\mathbf{p_1} + m)\gamma^\nu\right] Tr\left[(\mathbf{p_3} + M)\gamma_\mu(\mathbf{p_4} - M)\gamma_\nu\right]$$

Trace Theorems[7]

The full set of trace theorems, including those involving $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, are:

1. $\mathrm{Tr}(I) = 4$
2. The trace of any odd number of γ-matrices is zero
3. $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$
4. $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho}$
5. The trace of γ^5 multiplied by an odd number of γ-matrices is zero
6. $\mathrm{Tr}(\gamma^5) = 0$
7. $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$
8. $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4i\epsilon^{\mu\nu\rho\sigma}$, where $\epsilon^{\mu\nu\rho\sigma}$ is antisymmetric under the interchange of any two indices.

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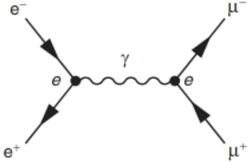
[7] M. Thomson, Modern Particle Physics, Section 6.5.3

The electron-positron annihilation



$$\langle |M_{fi}|^2 \rangle = \frac{e^4}{4q^4} Tr\left[(\mathbf{p_2} - m)\gamma^\mu(\mathbf{p_1} + m)\gamma^\nu\right] Tr\left[(\mathbf{p_3} + M)\gamma_\mu(\mathbf{p_4} - M)\gamma_\nu\right]$$

- I will be working in the limit where $m_e \approx 0$:

$$\langle |M_{fi}|^2 \rangle = \frac{e^4}{4q^4} Tr\left[\mathbf{p_2}\gamma^\mu\mathbf{p_1}\gamma^\nu\right] Tr\left[(\mathbf{p_3} + M)\gamma_\mu(\mathbf{p_4} - M)\gamma_\nu\right]$$

The electron-positron annihilation

$$
\begin{aligned}
Tr\left[\mathbf{p_2}\gamma^\mu \mathbf{p_1}\gamma^\nu\right] &= Tr\left[\gamma^\rho p_{2\rho}\gamma^\mu\gamma^\sigma p_{1\sigma}\gamma^\nu\right] = p_{2\rho}p_{1\sigma}\, Tr\left[\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\right] \\
&= 4p_{2\rho}p_{1\sigma}\left[g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}\right] \\
&= 4\left[p_{2\rho}g^{\rho\mu}p_{1\sigma}g^{\sigma\nu} - p_{2\rho}g^{\rho\sigma}p_{1\sigma}g^{\mu\nu} + p_{2\rho}g^{\rho\nu}p_{1\sigma}g^{\mu\sigma}\right] \\
&= 4\left[p_2^\mu p_1^\nu - g^{\mu\nu}(p_1\cdot p_2) + p_2^\nu p_1^\mu\right] \equiv 4\, Tr_1
\end{aligned}
$$

$$
Tr\left[(\mathbf{p_3}+M)\gamma_\mu(\mathbf{p_4}-M)\gamma_\nu\right] = Tr\left[(\mathbf{p_3}\gamma_\mu + M\gamma_\mu)(\mathbf{p_4}\gamma_\nu - M\gamma_\nu)\right]
$$

$$
= Tr\left[\mathbf{p_3}\gamma_\mu\mathbf{p_4}\gamma_\nu\right] - M\, Tr\left[\mathbf{p_3}\gamma_\mu\gamma_\nu\right] + M\, Tr\left[\gamma_\mu\mathbf{p_4}\gamma_\nu\right] - M^2\, Tr\left[\gamma_\mu\gamma_\nu\right]
$$

$$
= Tr\left[\mathbf{p_3}\gamma_\mu\mathbf{p_4}\gamma_\nu\right] - M^2\, Tr\left[\gamma_\mu\gamma_\nu\right]
$$

$$
= 4\left[p_{3\mu}p_{4\nu} - g_{\mu\nu}(p_3\cdot p_4) + p_{3\nu}p_{4\mu} - M^2 g_{\mu\nu}\right] \equiv 4\, Tr_2
$$

The electron-positron annihilation

$$
\begin{aligned}
Tr_1 \times Tr_2 &= [p_2^\mu p_1^\nu - g^{\mu\nu}(p_1 \cdot p_2) + p_2^\nu p_1^\mu] \\
&\times \left[p_{3\mu} p_{4\nu} - g_{\mu\nu}(p_3 \cdot p_4) + p_{3\nu} p_{4\mu} - M^2 g_{\mu\nu}\right] \\
&= p_2^\mu p_{3\mu} p_1^\nu p_{4\nu} - p_2^\mu p_1^\nu g_{\mu\nu}(p_3 \cdot p_4) + p_2^\mu p_{4\mu} p_1^\nu p_{3\nu} - M^2 p_2^\mu p_1^\nu g_{\mu\nu} \\
&- p_{3\mu} p_{4\nu} g^{\mu\nu}(p_1 \cdot p_2) + g^{\mu\nu} g_{\mu\nu}(p_1 \cdot p_2)(p_3 \cdot p_4) \\
&- g^{\mu\nu} p_{3\nu} p_{4\mu}(p_1 \cdot p_2) + M^2 g^{\mu\nu} g_{\mu\nu}(p_1 \cdot p_2) + p_2^\nu p_{4\nu} p_1^\mu p_{3\mu} \\
&- p_2^\nu p_1^\mu g_{\mu\nu}(p_3 \cdot p_4) + p_2^\nu p_{3\nu} p_1^\mu p_{4\mu} - M^2 p_2^\nu p_1^\mu g_{\mu\nu} \\
&= 2(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_2)(p_3 \cdot p_4)\,[4 - 4] \\
&+ 2(p_1 \cdot p_3)(p_2 \cdot p_4) + M^2(p_1 \cdot p_2)\,[4 - 2]
\end{aligned}
$$

$$
\Leftrightarrow\; Tr_1 \times Tr_2 = 2(p_1 \cdot p_4)(p_2 \cdot p_3) + 2(p_1 \cdot p_3)(p_2 \cdot p_4) + 2M^2(p_1 \cdot p_2)
$$

The electron-positron annihilation

$$\langle |M_{fi}|^2 \rangle = \frac{e^4}{4q^4} Tr\left[\mathbf{p_2}\gamma^\mu \mathbf{p_1}\gamma^\nu\right] Tr\left[(\mathbf{p_3}+M)\gamma_\mu(\mathbf{p_4}-M)\gamma_\nu\right]$$
$$= \frac{e^4}{4q^4} 4\,Tr_1 4\,Tr_2 = \frac{4e^4}{q^4} Tr_1 \times Tr_2$$

But $q^2 = (p_1+p_2)^2 = m_e^2 + m_e^2 + 2(p_1 \cdot p_2) \approx 2(p_1 \cdot p_2)$, therefore:

$$\langle |M_{fi}|^2 \rangle = \frac{4e^4}{4(p_1 \cdot p_2)^2} Tr_1 \times Tr_2$$
$$= 2e^4 \frac{(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) + M^2(p_1 \cdot p_2)}{(p_1 \cdot p_2)^2}$$

What If It Is Polarized???

Lets consider the case where both the electron beam and the positron beam can be polarized then, in the calculation of $\langle |M|^2 \rangle$ we must consider a weight factor in the sum.

$$\langle |M_{fi}|^2 \rangle = \sum_{spins} w_{spins} |M|^2$$

Where $w_{spins} = \alpha_i \beta_j$ and $\alpha_{R/L} = \frac{1 \pm P_e}{2}$; $\beta_{R/L} = \frac{1 \pm P_{e^+}}{2}$. I am considering $P = \frac{N_R - N_L}{N_R + N_L}$.

$$\langle |M|^2 \rangle = \cdots$$

$$= \frac{e^4}{q^4} \left[\left[\sum_{r=1}^{2} \beta_r \nu_m^r(p_2) \bar{\nu}_j^r(p_2) \right] \gamma_{ji}^{\mu} \left[\sum_{s=1}^{2} \alpha_s u_i^s(p_1) \bar{u}_n^s(p_1) \right] \gamma_{nm}^{\nu} \right] \mathcal{L}_{\mu\nu}^{(\mu)}$$

END