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From Fermi's golden rule to cross sections

High Energy Physics

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Intro

Introduction

Much of particle physics is based on experimental measurements of particle decay rates and interaction cross sections. These observable phenomena correspond to transitions between different quantum mechanical states. In non-relativistic quantum mechanics, transition rates are calculated using Fermi's golden rule, which I will derive next.

Derivation of Fermi's golden rule

Fermi's golden rule

Let $\phi_k(\boldsymbol{x}, t)$ denote the normalized solutions to the Schrödinger equation corresponding to the unperturbed, time-independent Hamiltonian $\hat{H}_0$, where

$$\hat{H}_0\phi_k = E_k\phi_k \qquad\qquad \langle\phi_j|\phi_k\rangle = \delta_{ij}$$

When an interaction Hamiltonian $\hat{H}'(\boldsymbol{x}, t)$ is introduced, capable of inducing transitions between states, the time-dependent Schrödinger equation takes the form:

$$i\frac{d\psi}{dt} = \left[\hat{H}_0 + \hat{H}'(\boldsymbol{x}, t)\right]\psi$$

Fermi's golden rule

The wavefunction $\psi(\boldsymbol{x}, t)$ can be written as a linear combination of the eigenstates of the unperturbed Hamiltonian:

$$\psi(\boldsymbol{x}, t) = \sum_k c_k(t)\phi_k(\boldsymbol{x})e^{-iE_k t}$$

The time-dependent coefficients $c_k(t)$ account for the transitions between states induced by the interaction.

Fermi's golden rule

Taking the wavefunction's time derivative,

$$
\begin{aligned}
i\frac{d\psi}{dt} &= i\sum_k \left[\frac{dc_k}{dt}\phi_k e^{-iE_k t} - ic_k E_k \phi_k e^{-iE_k t}\right] \\
&= \sum_k c_k E_k \phi_k e^{-iE_k t} + i\sum_k \frac{dc_k}{dt}\phi_k e^{-iE_k t} \\
&= \sum_k \hat{H}_0 c_k \phi_k e^{-iE_k t} + i\sum_k \frac{dc_k}{dt}\phi_k e^{-iE_k t}
\end{aligned}
$$

We then get,

$$
\sum_k \hat{H}' c_k(t)\phi_k(\boldsymbol{x})e^{-iE_k t} = i\sum_k \frac{dc_k}{dt}\phi_k e^{-iE_k t}
$$

Fermi's golden rule

Assume that at time $t = 0$, the initial wavefunction is $|i\rangle = \phi_i$, and the coefficients satisfy $c_k(0) = \delta_{ik}$. If the perturbing Hamiltonian, assumed to be constant for $t > 0$, is sufficiently weak such that $c_i(t) \approx 1$ and $c_{k\neq i}(t) \approx 0$ for all times, then, to first order, the previous equation can be approximated as:

$$i \sum_k \frac{dc_k}{dt} \phi_k e^{-iE_k t} \approx \hat{H}' \phi_i e^{-iE_i t}$$

Fermi's golden rule

The differential equation for the coefficient $c_f(t)$, corresponding to transitions to a particular final state $|f\rangle = \phi_f$, is derived by taking the inner product of both sides of the previous equation with $\phi_f(\boldsymbol{x})$ and using the orthogonality condition:

$$\langle f|i\sum_k \frac{dc_k}{dt}|k\rangle e^{-iE_k t} = \langle f|\hat{H}'|i\rangle e^{-iE_i t}$$

$$i\sum_k \frac{dc_k}{dt} e^{-iE_k t}\langle f|k\rangle = \langle f|\hat{H}'|i\rangle e^{-iE_i t}$$

$$i\frac{dc_f}{dt} e^{-iE_f t} = \langle f|\hat{H}'|i\rangle e^{-iE_i t}$$

Fermi's golden rule

The differential equation for the coefficient is

$$\frac{dc_f}{dt} = -i\langle f|\hat{H}'|i\rangle e^{i(E_f - E_i)t}$$

where

$$T_{fi} \equiv \langle f|\hat{H}'|i\rangle = \int \phi_f^*(\boldsymbol{x})\hat{H}'\phi_i(\boldsymbol{x})d^3\boldsymbol{x},$$

is the transition matrix element and has dimensions of energy.

Fermi's golden rule

At time $t = T$, the amplitude for transitions to the state $|f\rangle$ is given by the integral

$$c_f(T) = -i \int_0^T T_{fi} e^{i(E_f - E_i)t} dt$$

If the perturbing Hamiltonian is time-independent, the term $\langle f|\hat{H}'|i\rangle$ is also time-independent, and thus

$$c_f(T) = -i T_{fi} \int_0^T e^{i(E_f - E_i)t} dt$$

Fermi's golden rule

The probability of a transition to the state $|f\rangle$ is given by

$$P_{fi} = c_f(T)c_f^*(T) = |T_{fi}|^2 \int_0^T \int_0^T e^{i(E_f - E_i)t} e^{-i(E_f - E_i)t'} \, dt \, dt'$$

The transition rate $d\Gamma_{fi}$ from the initial state $|i\rangle$ to the specific final state $|f\rangle$ is therefore

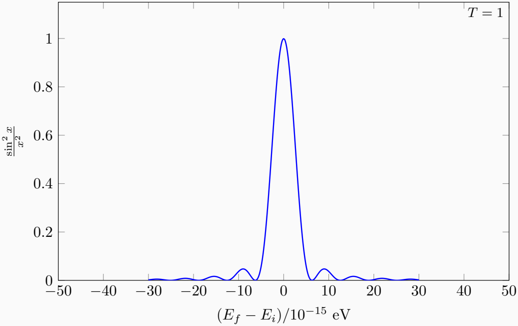
$$\Gamma_{fi} = \frac{P_{fi}}{T} = \frac{1}{T} |T_{fi}|^2 \int_{-T/2}^{+T/2} \int_{-T/2}^{+T/2} e^{i(E_f - E_i)t} e^{-i(E_f - E_i)t'} \, dt \, dt'$$

where the limits of integration are shifted by the substitutions $t \to t + \frac{T}{2}$ and $t' \to t' + \frac{T}{2}$.

Fermi's golden rule

The exact solution to the previous integral takes the form

$$\frac{\sin^2 x}{x^2} \qquad\qquad x = \frac{(E_f - E_i)T}{2\hbar}$$



It is evident that the transition rate is substantial only for final states where $E_f \approx E_i$ and that energy is conserved within the limits of the energy–time uncertainty relation ($\Delta E \Delta t \sim \hbar$).

Fermi's golden rule

The narrowness of the functional form of this last equation means that for all practical purposes, it can be written as

$$\Gamma_{fi} = |T_{fi}|^2 \lim_{T\to\infty} \left\{ \frac{1}{T} \int_{-T/2}^{+T/2} \int_{-T/2}^{+T/2} e^{i(E_f - E_i)t} e^{-i(E_f - E_i)t'} dt\, dt' \right\}$$

Using the definition of the Dirac delta-function the integral over dt' can be replaced by $2\pi\delta(E_f - E_i)$

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \lim_{T\to\infty} \left\{ \frac{1}{T} \int_{-T/2}^{+T/2} e^{i(E_f - E_i)t} \delta(E_f - E_i) dt \right\}$$

Fermi's golden rule

If there are *dn* accessible final states within the energy range $E_f \to E_f + dE_f$, then the total transition rate Γ_{fi} is given by

$$\begin{aligned}
\Gamma_{fi} &= 2\pi \int |T_{fi}|^2 \frac{dn}{dE_f} \lim_{T\to\infty} \left\{ \frac{1}{T} \int_{-T/2}^{+T/2} e^{i(E_f - E_i)t} \delta(E_f - E_i) dt \right\} dE_f \\
&= 2\pi \int |T_{fi}|^2 \frac{dn}{dE_f} \delta(E_f - E_i) \lim_{T\to\infty} \left\{ \frac{1}{T} \int_{-T/2}^{T/2} dt \right\} dE_f \\
&= 2\pi \int |T_{fi}|^2 \frac{dn}{dE_f} \delta(E_f - E_i) dE_f \\
&= 2\pi |T_{fi}|^2 \left| \frac{dn}{dE_f} \right|_{E_i}
\end{aligned}$$

This last term is referred to as the density of states, and is often written as $\rho(E_i)$.

Fermi's golden rule

Fermi's golden rule for the total transition rate is thus given by

$$\boxed{\Gamma_{fi} = 2\pi \left| T_{fi} \right|^2 \rho(E_i)}$$

where, to first order, $T_{fi} = \langle f | \hat{H}' | i \rangle$. If we wanted to take this to the second order,

$$T_{fi} = \langle f | \hat{H}' | i \rangle + \sum_{k \neq i} \frac{\langle f | \hat{H}' | k \rangle \langle k | \hat{H}' | i \rangle}{E_i - E_k}.$$

The transition rate between two states depends on the transition matrix element, which contains the fundamental particle physics, and the density of final states, which depends on the kinematics of the process being considered.

Phase space and wavefunction normalisation

Phase space and wavefunction normalisation

So far, we have been discussing the non-relativistic treatment of Fermi's golden rule and before discussing the relativistic treatment we will discuss the decay rate for the process $a \to 1 + 2$. To first order in perturbation theory, the transition matrix element is

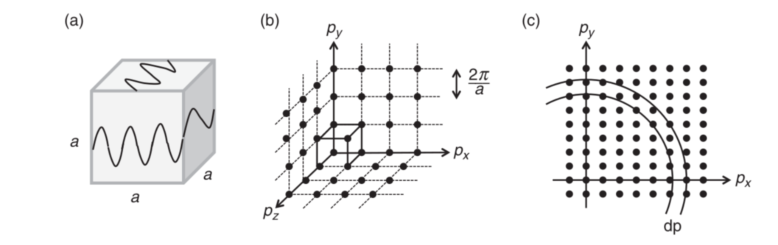
$$T_{fi} = \langle \psi_1 \psi_2 | \hat{H}' | \psi_a \rangle = \int \psi_1^* \psi_2^* \hat{H}' \psi_a^* d^3\boldsymbol{x}$$

In the Born approximation, the perturbation is assumed to be weak, and both the initial and final state particles are described by plane waves of the form

$$\psi(\boldsymbol{x}, t) = A e^{i(\boldsymbol{p}\cdot\boldsymbol{x} - Et)}$$

Phase space and wavefunction normalisation

It is standard practice to adopt a normalization scheme in which each plane wave corresponds to a single particle confined within a cubic volume of side length a.



Phase space and wavefunction normalisation

This scheme implies that

- the normalisation constant is

$$A^2 = \frac{1}{a^3} = \frac{1}{V}$$

- the wavefunction satisfies the periodic boundary conditions

$$\psi(x + a, y, z) = \psi(x, y, z)...$$

- The periodic boundary conditions on the wavefunction imply that the components of momentum are quantised

$$(p_x, p_y, p_z) = (n_x, n_y, n_z)\frac{2\pi}{a}$$

Phase space and wavefunction normalisation

This imposes a restriction on the allowed momentum states, limiting them to a discrete set as illustrated in figure b. Each state in momentum space occupies a cubic volume of

$$d^3\boldsymbol{p} = dp_x dp_y dp_z = \left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$$

The number of states dn with momentum magnitude in the range $p \to p + dp$ is given by the volume of the spherical shell in momentum space at radius p and thickness dp, divided by the average volume occupied by a single state, $(2\pi)^3/V$

$$dn = 4\pi p^2 dp \times \frac{V}{(2\pi)^3} \implies \frac{dn}{dp} = \frac{4\pi p^2}{(2\pi)^3} V$$

Phase space and wavefunction normalisation

The density of states appearing in Fermi's golden rule can then be determined from

$$\rho(E) = \frac{dn}{dE} = \frac{dn}{dp}\left|\frac{dp}{dE}\right|$$

The density of states represents the number of accessible momentum states for a given decay and increases with the momentum of the final-state particle. Therefore, all else being equal, decays to lighter particles, which are produced with larger momentum, are favored over decays to heavier particles.

Phase space and wavefunction normalisation

The calculation of the decay rate does not depend on the normalization volume, as the volume dependence in the phase space expression is canceled by the factors of V associated with the wavefunction normalization that appears in the square of the transition matrix element. Since the volume does not affect the final result, it is convenient to normalize to one particle per unit volume by setting $V = 1$. In this case, the number of accessible states for a particle associated with an infinitesimal volume in momentum space, $d^3\boldsymbol{p}_i$, is simply

$$dn_i = \frac{d^3\boldsymbol{p}_i}{(2\pi)^3}$$

Phase space and wavefunction normalisation

For the decay of a particle into a final state consisting of N particles, there are $N-1$ independent momenta in the final state. Therefore, the number of independent states for an N-particle final state is

$$dn = \prod_{i=1}^{N-1} dn_i = \prod_{i=1}^{N-1} \frac{d^3\boldsymbol{p}_i}{(2\pi)^3}$$

This can be expressed in another form, by including the momentum space volume element for the N-th particle, $d^3\boldsymbol{p}_N$, and using a three-dimensional delta function to enforce momentum conservation

$$dn = (2\pi)^3 \prod_{i=1}^{N} \frac{d^3\boldsymbol{p}_i}{(2\pi)^3} \delta^3\left(\boldsymbol{p}_a - \sum_{i=1}^{N} \boldsymbol{p}_i\right)$$

where $\boldsymbol{p}_a$ is the momentum of the decaying particle.

Lorentz-invariant phase space

The wavefunction normalization of one particle per unit volume is not Lorentz invariant, as it applies only to a specific frame of reference. In a different reference frame, the original normalization volume will be Lorentz contracted by a factor of $1/\gamma$ along the direction of relative motion. Thus, the original normalization of one particle per unit volume corresponds to a normalization of $\gamma = E/m$ particles per unit volume in the boosted frame. Therefore, a Lorentz-invariant wavefunction normalization must be proportional to E particles per unit volume, such that the increase in energy compensates for the Lorentz contraction. The standard convention is to normalize to $2E$ particles per unit volume.

$$\int \psi^* \psi d^3\mathbf{x} = 1 \qquad \int \psi'^* \psi' d^3\mathbf{x} = 2E \qquad \psi' = \sqrt{2E}\psi$$

Lorentz-invariant phase space

For a general process, $a + b + \cdots \to 1 + 2 + \cdots$, the Lorentz-invariant matrix element is defined as

$$\mathcal{M}_{fi} = \langle \psi_1' \psi_2' \cdots | \hat{H}' | \psi_a' \psi_b' \cdots \rangle = \sqrt{2E_1 2E_2 \cdots 2E_a 2E_b \cdots}\, T_{fi}$$

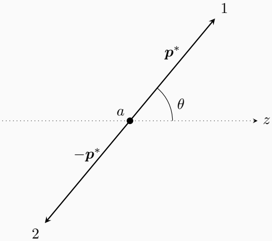
Back to Fermi's golden rule

For a two-body decay $a \to 1 + 2$, the quantum mechanical transition rate is given by Fermi's golden rule

$$
\begin{aligned}
\Gamma_{fi} &= 2\pi \int |T_{fi}|^2 \, \delta(E_a - E_1 - E_2) dn \\
&= (2\pi)^4 \int |T_{fi}|^2 \, \delta(E_a - E_1 - E_2) \delta^3 \left(\boldsymbol{p}_a - \boldsymbol{p}_1 - \boldsymbol{p}_2\right) \frac{d^3\boldsymbol{p}_1}{(2\pi)^3} \frac{d^3\boldsymbol{p}_2}{(2\pi)^3} \\
&= \frac{(2\pi)^4}{2E_a} \int |\mathcal{M}_{fi}|^2 \, \delta(E_a - E_1 - E_2) \delta^3 \left(\boldsymbol{p}_a - \boldsymbol{p}_1 - \boldsymbol{p}_2\right) \frac{d^3\boldsymbol{p}_1}{(2\pi)^3 2E_1} \frac{d^3\boldsymbol{p}_2}{(2\pi)^3 2E_2}
\end{aligned}
$$

The matrix element $\mathcal{M}_{fi}$ is defined in terms of wavefunctions with a Lorentz-invariant normalization, and the phase space integration elements, $d^3\boldsymbol{p}_i/E_i$, are also Lorentz invariant. Therefore, the integral is Lorentz invariant, and thus it represents Fermi's golden rule in a Lorentz-invariant form.

Particle decays



Consider the two-body decay $a \to 1 + 2$. In the centre-of-mass frame, the decaying particle is at rest, so $E_a = m_a$ and $\mathbf{p}_a = 0$, and the two daughter particles are produced back-to-back with three-momenta $\mathbf{p}^*$ and $-\mathbf{p}^*$. In this frame, the decay rate is given by

$$\Gamma_{fi} = \frac{1}{8\pi^2 m_a} \int |\mathcal{M}_{fi}|^2 \, \delta(m_a - E_1 - E_2)\delta^3(\boldsymbol{p}_1 + \boldsymbol{p}_2)\frac{d^3\boldsymbol{p}_1 \, d^3\boldsymbol{p}_2}{4E_1 E_2}$$

Example ii

After some manipulation and using $d^3\boldsymbol{p}_1 = p_1^2 dp_1 \sin\theta d\theta d\phi = p_1^2 dp_1 d\Omega$, the previous equation becomes

$$\Gamma_{fi} = \frac{p^*}{32\pi^2 m_a^2} \int |\mathcal{M}_{fi}|^2 \, d\Omega$$

This is the general expression for any two-body decay. The fundamental physics is contained in the matrix element and the additional factors arise from the phase space integral.

Cross sections

Interaction cross sections

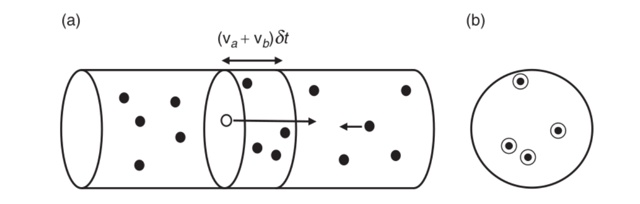
The calculation of interaction rates is slightly more complicated than that for particle decays because it is necessary to account for the flux (number of particles crossing a unit area per unit time) of initial-state particles.

In the simplest scenario, a beam of particles of type a with flux ϕ_a passes through a region containing n_b target particles of type b per unit volume. The interaction rate per target particle, r_b, is proportional to the flux and can be written as

$$r_b = \sigma \phi_a$$

The fundamental physics is in σ, which has dimensions of area and is referred to as the interaction cross section. In general, the cross section represents the underlying quantum mechanical probability that an interaction will occur.

Interaction cross sections



Consider a single incident particle of type a moving with velocity v_a through a region of area A that contains n_b particles of type b per unit volume, with b-particles moving in the opposite direction at velocity v_b. In a time interval δt, particle a traverses a region containing $\delta N = n_b(v_a + v_b)A\,\delta t$ particles of type b.

Interaction cross sections

The interaction probability can be obtained by dividing the effective total cross-sectional area of the δN particles by the area A.

$$\delta P = \frac{\delta N \sigma}{A} = n_b(v_a + v_b)\sigma\delta t = n_b v \sigma \delta t$$

Hence the interaction rate for each particle of type a is

$$r_a = \frac{dP}{dt} = n_b v \sigma$$

For a beam of particles of type a, with number density na confined to a volume V, the total interaction rate is

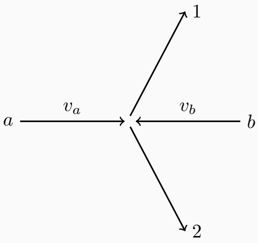
$$\begin{aligned} \text{rate} &= r_a n_a V = n_b v \sigma n_a V = \phi N_b \sigma \\ &= \text{flux} \times \# \text{ target particles} \times \text{cross section} \end{aligned}$$

Interaction cross sections

Thus,

$$\sigma = \frac{\text{number of interactions per unit time per target particle}}{\text{incident flux}}$$

Example



Consider the scattering process $a + b \to 1 + 2$, as observed in the rest frame, where the particles of type a have velocity $\boldsymbol{v}_a$ and the particles of type b have velocity $\boldsymbol{v}_b$.

Example

If the number densities of the particles are n_a and n_b, the interaction rate in the volume V is given by

$$\text{rate} = \phi_a n_b V\sigma = (v_a + v_b) n_a n_b V\sigma$$

Normalizing the wavefunctions to one particle in a volume V gives $n_a = n_b = 1/V$, for which the interaction rate in the volume V is

$$\Gamma_{fi} = \frac{v_a + v_b}{V}\sigma$$

It is once again convenient to adopt a normalization of one particle per unit volume. With this choice, the cross section is related to the transition rate by

$$\sigma = \frac{\Gamma_{fi}}{v_a + v_b}$$

Interaction cross sections

The transition rate Γ_{fi} is given by Fermi's golden rule, resulting in

$$\sigma = \frac{(2\pi)^4}{v_a + v_b} \int |T_{fi}|^2 \, \delta(E_a + E_b - E_1 - E_2) \delta^3(\boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_1 - \boldsymbol{p}_2) \frac{d^3\boldsymbol{p}_1}{(2\pi)^3} \frac{d^3\boldsymbol{p}_2}{(2\pi)^3}$$

or, in the Lorentz-invariant form,

$$\sigma = \frac{(2\pi)^{-2}}{4E_a E_b (v_a + v_b)} \int |\mathcal{M}_{fi}|^2 \, \delta(E_a + E_b - E_1 - E_2) \delta^3(\boldsymbol{p}_a + \boldsymbol{p}_b - \boldsymbol{p}_1 - \boldsymbol{p}_2) \frac{d^3\boldsymbol{p}_1}{2E_1} \frac{d^3\boldsymbol{p}_2}{2E_2}$$

Interaction cross sections

The most convenient frame choice is the centre-of-mass frame where $\boldsymbol{p}_a = -\boldsymbol{p}_b = \boldsymbol{p}_i^*$, $\boldsymbol{p}_1 = -\boldsymbol{p}_2 = \boldsymbol{p}_f^*$ and the centre-of-mass energy is $\sqrt{s} = (E_a^* + E_b^*)$. In this case the cross section takes the form

$$\sigma = \frac{(2\pi)^{-2}}{4p_i^*\sqrt{s}} \int |\mathcal{M}_{fi}|^2 \, \delta(\sqrt{s} - E_1 - E_2)\delta^3(\boldsymbol{p}_1 + \boldsymbol{p}_2)\frac{d^3\boldsymbol{p}_1}{2E_1}\frac{d^3\boldsymbol{p}_2}{2E_2}$$

After some manipulation,

$$\sigma = \frac{1}{64\pi^2 s}\frac{p_f^*}{p_i^*}\int |\mathcal{M}_{fi}|^2 \, d\Omega^*$$

This represents the cross section for any two-body $\rightarrow$ two-body process.

Summary

Summary i

1. We started by deriving Fermi's golden rule

$$\Gamma_{fi} = 2\pi \, |T_{fi}|^2 \, \rho(E_i)$$

and the density of states

$$\rho(E_i) = \left| \frac{dn}{dE_f} \right|_{E_i} \qquad dn = (2\pi)^3 \prod_{i=1}^{N} \frac{d^3\boldsymbol{p}_i}{(2\pi)^3} \delta^3 \left(\boldsymbol{p}_a - \sum_{i=1}^{N} \boldsymbol{p}_i \right)$$

and finally the Lorentz-invariant matrix element

$$\mathcal{M}_{fi} = \langle \psi_1' \psi_2' \cdots | \hat{H}' | \psi_a' \psi_b' \cdots \rangle = \sqrt{2E_1 2E_2 \cdots 2E_a 2E_b \cdots} T_{fi}$$

2. We then obtained the result for a 2-body decay ($a \to 1 + 2$)

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \, \delta(E_a - E_1 - E_2)\delta^3\left(\boldsymbol{p}_a - \boldsymbol{p}_1 - \boldsymbol{p}_2\right) \frac{d^3\boldsymbol{p}_1}{(2\pi)^3} \frac{d^3\boldsymbol{p}_2}{(2\pi)^3}$$
$$= \frac{(2\pi)^4}{2E_a} \int |\mathcal{M}_{fi}|^2 \, \delta(E_a - E_1 - E_2)\delta^3\left(\boldsymbol{p}_a - \boldsymbol{p}_1 - \boldsymbol{p}_2\right) \frac{d^3\boldsymbol{p}_1}{(2\pi)^3 2E_1} \frac{d^3\boldsymbol{p}_2}{(2\pi)^3 2E_2}$$

3. and for a 2-body $\to$ 2-body scattering process ($a + b \to 1 + 2$)

$$\sigma = \frac{(2\pi)^4}{v_a + v_b} \int |T_{fi}|^2 \, \delta(E_{in} - E_{out})\delta^3(\boldsymbol{p}_{in} - \boldsymbol{p}_{out}) \frac{d^3\boldsymbol{p}_1}{(2\pi)^3} \frac{d^3\boldsymbol{p}_2}{(2\pi)^3}$$
$$= \frac{(2\pi)^{-2}}{4E_a E_b(v_a + v_b)} \int |\mathcal{M}_{fi}|^2 \, \delta(E_{in} - E_{out})\delta^3(\boldsymbol{p}_{in} - \boldsymbol{p}_{out}) \frac{d^3\boldsymbol{p}_1}{2E_1} \frac{d^3\boldsymbol{p}_2}{2E_2}$$

Summary iii

4. After that we've done a few examples
   4.1 For a 2-body ($a \to 1+2$) decay in the rest frame of a
   $$\Gamma_{fi} = \frac{p^*}{32\pi^2 m_a^2} \int |\mathcal{M}_{fi}|^2 \, d\Omega$$
   4.2 For a 2-body $\to$ 2-body ($a + b \to 1 + 2$) scattering in the collision centre of mass frame
   $$\sigma = \frac{1}{64\pi^2 s} \frac{p_f^*}{p_i^*} \int |\mathcal{M}_{fi}|^2 \, d\Omega^*$$

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