Time ordered perturbation theory to Feynman diagrams

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Why perturbation theory?

Simple world ☺

The full Hamiltonian of the problem is solvable; in other words, one can find the eigenstates and the corresponding energies of those states.

$$\hat{H} = \hat{H}_0$$

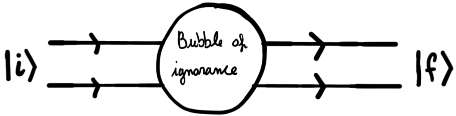
Simple to the point is boring... transitions between different energy eigenstates do not occur because the Hamiltonian is interaction free, and the system evolves "constant" in time.

The initial and final states are identical, reflecting the absence of interactions.

Why perturbation theory?

Real world ☹

Now the Hamiltonian includes interactions: $\hat{H} = \hat{H}_0 + \hat{V}$.



The transition rate $\Gamma_{i\to f}$ between an initial state $|i\rangle$ and a final state $|f\rangle$ is given by Fermi's golden rule

$$\Gamma_{i\to f} = 2\pi |T_{fi}|^2 \rho(E_f)$$

where T_{fi} is the transition matrix element, given by the perturbation expansion

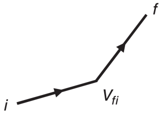
$$T_{fi} = \langle f|V|i\rangle + \sum_{j\neq i} \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} + \cdots$$

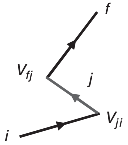
j is the intermediate state.

How the interaction potential acts?

$$T_{fi} = \langle f|V|i\rangle + \sum_{j\neq i} \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} + \cdots$$

The first two terms in the perturbation series can be viewed as "scattering in a potential" and "scattering via an intermediate state j".





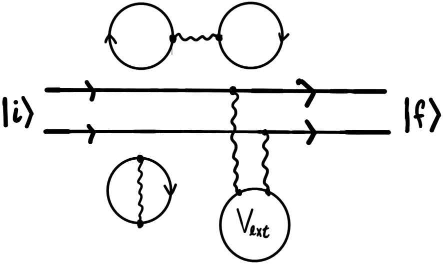
In Quantum Field Theory, interactions between particles are mediated by the exchange of other particles and there is no mysterious action at a distance. The forces between particles result from the transfer of the momentum carried by the exchanged particle.

In QFT language

The first order can be ignored...

$$\langle f|V|i\rangle = 0.$$

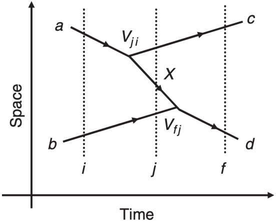
Diagrams without external legs (vacuum bubbles) correspond to vacuum-to-vacuum amplitudes.

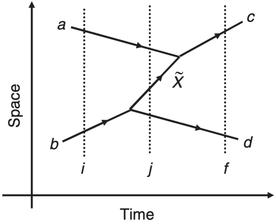


External field interactions shift the energy, but since only energy differences matter, they don't contribute to measurable interactions.

Time-ordered perturbation theory

Considering the particle interaction $a + b \to c + d$, which can occur via an intermediate state corresponding to the exchange of **one** particle X. The process may proceed in 2 ways:





- Particle a emits particle X which is later absorbed by particle b:

$$|i\rangle \equiv a + b, \quad |j\rangle \equiv c + X + b, \quad |f\rangle \equiv c + d \quad \textbf{(left)}$$

- Particle b emits particle $\tilde{X}$ which is later absorbed by particle a:

$$|i\rangle \equiv a + b, \quad |j\rangle \equiv a + \tilde{X} + d, \quad |f\rangle \equiv c + d \quad \textbf{(right)}$$

2 Time-Orders

The full calculation must include all possible time orderings to ensure that the final result is **Lorentz invariant** (independent of the observer's frame of reference).

- **Left**: $|j\rangle \equiv c + X + b$

$$T_{fi}^{ab} = \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} = \frac{\langle d|V|b+X\rangle\langle c+X|V|a\rangle}{(E_a+E_b)-(E_c+E_X+E_b)}$$

- **Right**: $|j\rangle \equiv a + \tilde{X} + d$

$$T_{fi}^{ba} = \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} = \frac{\langle c|V|a+\tilde{X}\rangle\langle \tilde{X}+d|V|b\rangle}{(E_a+E_b)-(E_a+E_{\tilde{X}}+E_d)}$$

- **Left**: $|j\rangle \equiv c + X + b$

$$T_{fi}^{ab} = \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} = \frac{\langle d|V|b+X\rangle\langle c+X|V|a\rangle}{(E_a+E_b)-(E_c+E_X+E_b)}$$

The interactions at the two vertices are defined by the non-invariant matrix elements

$$V_{ji} = \langle c+X|V|a\rangle, \quad V_{fj} = \langle d|V|X+b\rangle.$$

The non-invariant matrix elements are related to the Lorentz-invariant matrix elements by

$$V_{ji} = \frac{\mathcal{M}_{a\to c+X}}{(2E_a 2E_c 2E_X)^{1/2}}, \quad V_{fj} = \frac{\mathcal{M}_{b+X\to d}}{(2E_b 2E_d 2E_X)^{1/2}}.$$

Assuming the simplest possible Lorentz-invariant coupling, namely a scalar:

$$V_{ji} = \frac{g_a}{(2E_a 2E_c 2E_X)^{1/2}}, \quad V_{fj} = \frac{g_b}{(2E_b 2E_d 2E_X)^{1/2}}.$$

Therefore, the second-order term in the perturbation series is

$$T_{fi}^{ab} = \frac{1}{2E_X} \cdot \frac{1}{(2E_a 2E_b 2E_c 2E_d)^{1/2}} \cdot \frac{g_a g_b}{(E_a - E_c - E_X)}.$$

The LI matrix element for this time order is then

$$\mathcal{M}_{fi}^{ab} = (2E_a 2E_b 2E_c 2E_d)^{1/2}\, T_{fi}^{ab} = \frac{1}{2E_X} \cdot \frac{g_a g_b}{(E_a - E_c - E_X)}.$$

Notes:

- $\mathcal{M}_{fi}^{ab}$ is **not** Lorentz-invariant - order of events depends on frame.
- Momentum is conserved at each vertex, but energy **is not** ($E_j \neq E_i$).
- Particle X is **on mass shell**.

Repeating the same steps for the other time ordering yields...

$$\mathcal{M}_{fi}^{ba} = (2E_a 2E_b 2E_c 2E_d)^{1/2}\, T_{fi}^{ba} = \frac{1}{2E_X} \cdot \frac{g_a g_b}{(E_b - E_d - E_X)}.$$

The total amplitude is given by the sum of the two time-ordered amplitudes

$$\mathcal{M}_{fi} = \mathcal{M}_{fi}^{ab} + \mathcal{M}_{fi}^{ba} = \frac{g_a g_b}{2E_X} \cdot \left(\frac{1}{E_a - E_c - E_X} + \frac{1}{E_b - E_d - E_X} \right),$$

which, using energy conservation $E_b - E_d = E_c - E_a$, can be written

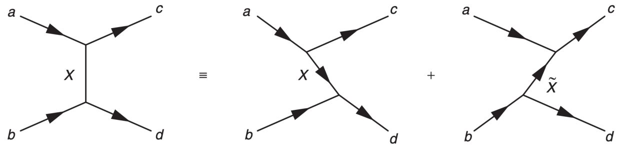
$$\mathcal{M}_{fi} = \frac{g_a g_b}{2E_X} \cdot \left(\frac{1}{E_a - E_c - E_X} - \frac{1}{E_a - E_c + E_X} \right) = \frac{g_a g_b}{(E_a - E_c)^2 - E_X^2}.$$

Propagator

By the relativistic dispersion relation and momentum conservation,

$$\mathcal{M}_{fi} = \frac{g_a g_b}{q^2 - m_X^2}$$

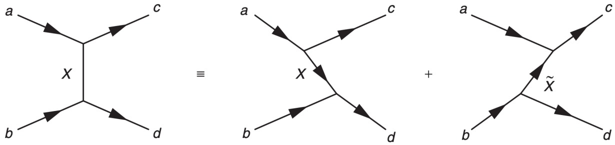
where $q = p_a - p_c$ is the momentum of the exchanged virtual particle X.



The sum over all possible time-orderings is represented by a Feynman diagram. Both momentum and energy are conserved at the interaction vertices of a Feynman diagram.

Conclusion

$$\boxed{\mathcal{M}_{fi} = \frac{g_a g_b}{q^2 - m_X^2}}$$



Feynman diagram

- Momentum **and** energy are conserved in vertices.
- Exchanged particle is **off mass shell**.
- X is a **virtual** particle.

Time-ordered

- Momentum is conserved in vertices.
- But energy is not conserved.
- Exchanged particle is **on mass shell**.

References

[1] M. Thompson, *Modern Particle Physics*, Cambridge University Press, 2013.

[2] R. Gonçalo, *High Energy Physics: Interactions by Particle Exchange*, Lecture Slides, 2025.