QED: calculation of matrix elements from spin sums

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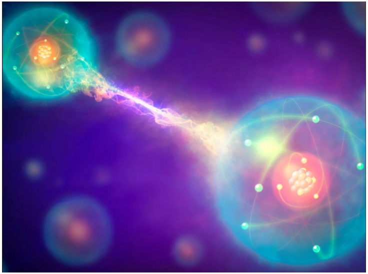

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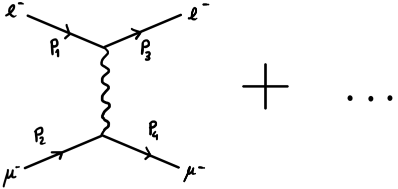
QED Calculations

How to calculate a cross section using QED (e.g $e^-\mu^- \to e^-\mu^-$):

1. Draw all possible Feynman Diagrams
   - For $e^-\mu^- \to e^-\mu^-$ there is just one lowest order diagram+ many second order diagrams $+ \ldots$



2. For each diagram calculate the matrix element using Feynman rules
3. Sum the individual matrix elements (i.e. sum the amplitudes)

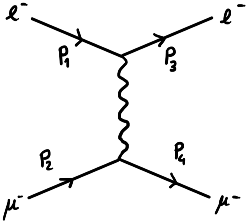
$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \ldots$$

- Note: summing amplitudes may interfere constructively or destructively!

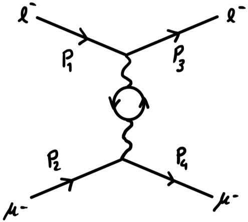
4. Square the matrix element (this gives the full perturbation expansion in α_{em}):

$$|\mathcal{M}_{fi}|^2 = (\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \dots)(\mathcal{M}_1^* + \mathcal{M}_2^* + \mathcal{M}_3^* + \dots)$$

   - For QED $\alpha_{em} \sim 1/137$ so the lowest order diagram dominates and higher orders can often be neglected



$|\mathcal{M}|^2 \propto \alpha_{em}^2$



$|\mathcal{M}|^2 \propto \alpha_{em}^4$

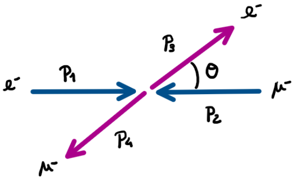
5. Calculate decay rate/cross section using formulas we have derived in previous classes

$e^-\mu^- \to e^-\mu^-$ Scattering

★ Consider the process: $e^-\mu^- \to e^-\mu^-$

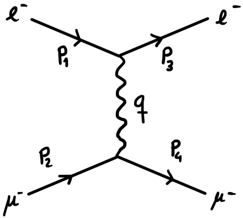
- In the center-of-mass ($E \gg m$) frame:

$$p_1 = (E, 0, 0, p) \qquad p_2 = (E, 0, 0, -p)$$
$$p_3 = (E, \vec{p}_f) \qquad p_4 = (E, -\vec{p}_f)$$



- Only consider the lowest order Feynman diagram, the Lorentz-invariant Matrix Element is:

$$\mathcal{M}_{fi} = -\frac{e^2}{q^2}\left[\bar{u}_{p_3}\gamma^{\mu}u_{p_1}\right]g_{\mu\nu}\left[\bar{u}_{p_4}\gamma^{\nu}u_{p_2}\right]$$



Electron and Muon Currents

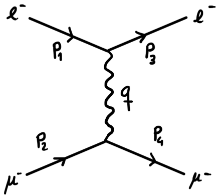
- The matrix element can be written in terms of the electron and muon currents :

$$\mathcal{M}_{fi} = -\frac{e^2}{q^2} j^{\mu}_{(e)} g_{\mu\nu} j^{\nu}_{(\mu)}$$

where $j^{\mu}_{(e)} = \bar{u}_{p_3}\gamma^{\mu}u_{p_1}$ and $j^{\nu}_{(\mu)} = \bar{u}_{p_4}\gamma^{\nu}u_{p_2}$ are 4-vector currents.

- The photon momentum is $q = p_3 - p_1$ giving $q^2 = (p_3 - p_1)^2 = t$

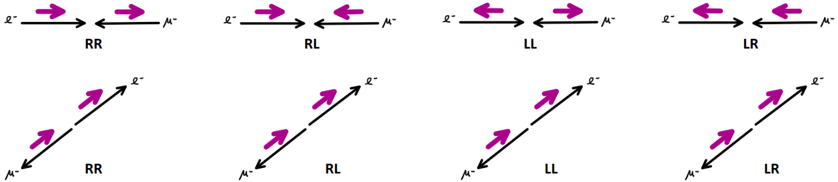
$$\mathcal{M}_{fi} = -\frac{e^2}{t} j^{\mu}_{(e)} g_{\mu\nu} j^{\nu}_{(\mu)}$$



- Matrix element is a four-vector scalar product, confirming it is Lorentz Invariant

Spin in $e^-\mu^- \to e^-\mu^-$ Scattering

- To calculate the $e^-\mu^- \to e^-\mu^-$ cross section, the matrix element of needs to be evaluated considering the possible spin states of the particles involved.
- There are four possible helicity configurations in the initial state, and four possible helicity configurations in the final state. :



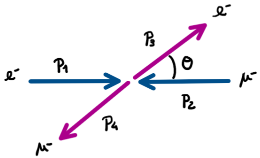
- In total there are 16 combinations e.g. RL→RR, RL→RL, ...

$$\langle |M|^2 \rangle = \frac{1}{4} \sum_{\text{Spins}} |M_i|^2 = \frac{1}{4} \left(|M_{LL\to LL}|^2 + |M_{LL\to LR}|^2 + \dots \right)$$

Helicity States in C.o.M. Frame

- In the C.o.M. frame ($E \gg m$):

$$p_1 = (E, 0, 0, E); \quad p_2 = (E, 0, 0, -E)$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta);$$
$$p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$



- Left-and right-handed helicity spinors for particles/anti-particles are:

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix} \quad u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix} \quad v_\uparrow = \sqrt{E}\begin{pmatrix} \frac{|\vec{p}|}{E+m}s \\ \frac{|\vec{p}|}{E+m}e^{i\phi}c \\ -s \\ -e^{i\phi}c \end{pmatrix} \quad v_\downarrow = \sqrt{E}\begin{pmatrix} \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \\ -c \\ -e^{i\phi}s \end{pmatrix}$$

where $s = \sin(\theta/2)$, $c = \cos(\theta/2)$

- In the high energy limit $E \gg m$:

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix} \quad u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix} \quad v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix} \quad v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ -c \\ -se^{i\phi} \end{pmatrix}$$

- The initial-state e^- can either be in a left- or right-handed helicity state $(\theta = 0, \phi = 0)$ and for the initial state μ^- $(\theta = \pi, \phi = \pi)$ we have respectively:

$$u_\uparrow(p_1) = \sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \quad u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} \quad u_\uparrow(p_2) = \sqrt{E}\begin{pmatrix}0\\-1\\0\\-1\end{pmatrix} \quad u_\downarrow(p_2) = \sqrt{E}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}$$

- Similarly for the final state of e^- $(\theta,\ \phi = 0)$ and μ^- $(\theta \to \pi - \theta,$ $\phi \to \pi)$:

$$u_\uparrow(p_3) = \sqrt{E}\begin{pmatrix}c\\s\\c\\s\end{pmatrix} \quad u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix}-s\\c\\s\\-c\end{pmatrix} \quad u_\uparrow(p_4) = \sqrt{E}\begin{pmatrix}s\\-c\\s\\-c\end{pmatrix} \quad u_\downarrow(p_4) = \sqrt{E}\begin{pmatrix}-c\\-s\\c\\s\end{pmatrix}$$

- Where $s = \sin(\theta/2)$, $c = \cos(\theta/2)$ and we use identities:

$$\sin\left(\frac{\pi - \theta}{2}\right) = \cos\frac{\theta}{2}, \quad \cos\left(\frac{\pi - \theta}{2}\right) = \sin\frac{\theta}{2}, \quad e^{i\pi} = -1$$

The electron Current

- We want to evaluate $(j_{(e)})^\nu = \bar{u}(p_3)\gamma^\nu u(p_1)$ for all 4 helicity combinations.
- For arbitrary spinors ψ, ϕ, it can be shown that the components of $\bar{\psi}\gamma^\mu\phi$ are:

$$
\begin{aligned}
\bar{\psi}\gamma^0\phi &= \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 \\
\bar{\psi}\gamma^1\phi &= \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 \\
\bar{\psi}\gamma^2\phi &= -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) \\
\bar{\psi}\gamma^3\phi &= \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2
\end{aligned}
$$

- Consider the $e_R^- e_R^-$ combination using $\psi = u_\uparrow(p_3)$, $\phi = u_\uparrow(p_1)$

$$
u_\uparrow(p_3) = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix} \quad u_\uparrow(p_1) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{cases} \bar{u}_\uparrow(p_3)\gamma^0 u_\downarrow(p_1) = 2E\cos(\theta/2), \\ \bar{u}_\uparrow(p_3)\gamma^1 u_\downarrow(p_1) = 2E\sin(\theta/2), \\ \bar{u}_\uparrow(p_3)\gamma^2 u_\downarrow(p_1) = 2iE\sin(\theta/2), \\ \bar{u}_\uparrow(p_3)\gamma^3 u_\downarrow(p_1) = 2E\cos(\theta/2). \end{cases}
$$

So, the four-vector eletron current for the RR combination is:

$$j_{(e)}^{RR} = \overline{u}_{\uparrow}(p_3)\gamma^{\nu} u_{\uparrow}(p_1) = 2E(\cos(\theta/2), \sin(\theta/2), i\sin(\theta/2), \cos(\theta/2))$$

Doing the same for the other 4 helicity combinations we obtain:

$$\begin{aligned} j_{(e)}^{RR} &= \overline{u}_{\uparrow}(p_3)\gamma^{\nu} u_{\uparrow}(p_1) = 2E(c, s, is, c) \\ j_{(e)}^{LL} &= \overline{u}_{\downarrow}(p_3)\gamma^{\nu} u_{\downarrow}(p_1) = 2E(c, s, -is, c) \\ j_{(e)}^{RL} &= \overline{u}_{\uparrow}(p_3)\gamma^{\nu} u_{\downarrow}(p_1) = (0, 0, 0, 0) \\ j_{(e)}^{LR} &= \overline{u}_{\downarrow}(p_3)\gamma^{\nu} u_{\uparrow}(p_1) = (0, 0, 0, 0) \end{aligned}$$

where $s = \sin(\theta/2)$, $c = \cos(\theta/2)$

- In the limit where $E \gg m$ only two helicity combinations are non-zero!
- This is an important feature of QED.
- From 16 possible helicity combinations only 4 give non-zero matrix elements

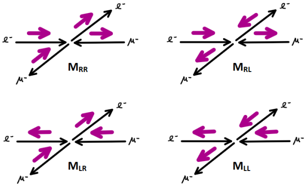
The muon current

- Recall that for the initial and final μ^- state we have $(\theta = \pi,\ \phi = \pi)$ and $(\theta \to \pi - \theta,\ \phi \to \pi)$:

$$u_\uparrow(p_2) = \sqrt{E}\begin{pmatrix}0\\-1\\0\\-1\end{pmatrix} \quad u_\downarrow(p_2) = \sqrt{E}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix} \quad u_\uparrow(p_4) = \sqrt{E}\begin{pmatrix}s\\-c\\s\\-c\end{pmatrix} \quad u_\downarrow(p_4) = \sqrt{E}\begin{pmatrix}-c\\-s\\c\\s\end{pmatrix}$$

- We can obtain $j_{(\mu)}$ just like we did for $j_{(e)}$ and we are left with:

$$j^{RR}_{(\mu)} = \overline{u}_\uparrow(p_3)\gamma^\nu u_\uparrow(p_1) = 2E(c, -s, is, -c)$$
$$j^{LL}_{(\mu)} = \overline{u}_\downarrow(p_3)\gamma^\nu u_\downarrow(p_1) = 2E(c, -s, -is, -c)$$
$$j^{RL}_{(\mu)} = \overline{u}_\uparrow(p_3)\gamma^\nu u_\downarrow(p_1) = (0, 0, 0, 0)$$
$$j^{LR}_{(\mu)} = \overline{u}_\downarrow(p_3)\gamma^\nu u_\uparrow(p_1) = (0, 0, 0, 0)$$



Matrix Element Calculation

- We can now calculate $\mathcal{M}_{fi} = -\frac{e^2}{t} j_{(e)}.j_{(\mu)}$ for the 4 possible helicity combinations.
- e.g the matrix element for $e^-_R \mu^-_R \to e^-_R \mu^-_R$:

$$
\begin{aligned}
\mathcal{M}_{RR\to RR} = \mathcal{M}_{RR} &= -\frac{e^2}{t} j^{RR}_{(e)} j^{RR}_{(\mu)} \\
&= -\frac{e^2}{t} \left[2E(c, s, is, c)\right] \cdot \left[2E(c, -s, is, -c)\right] \\
&= -\frac{e^2}{t} 4E^2 \left[c^2 + s^2 + s^2 + c^2\right] \quad \text{(Minkowski dot product)} \\
&= -\frac{e^2}{t} 4E^2 \times 2 \quad (\text{since } \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1) \\
&= -\frac{e^2}{t} 2s \quad (\; s = (2E)^2 \text{ in c.o.m. frame})
\end{aligned}
$$

- Similarly: $|\mathcal{M}_{RR}|^2 = |\mathcal{M}_{LL}|^2 = \left(-\frac{e^2}{t} 2s\right)^2$

$$|\mathcal{M}_{RL}|^2 = |\mathcal{M}_{LR}|^2 = \left(-\frac{e^2}{t} 2s\right)^2 \cos^2\frac{\theta}{2}$$

Differential Cross Section

- The spin-averaged matrix element for the process $e^-\mu^- \to e^-\mu^-$ is given by: $\langle|M_{fi}|^2\rangle = \frac{1}{4} \times \left(|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2 + |M_{LL}|^2\right)$
- The cross section is obtained by averaging over initial and summing over final spin states:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{p_f^*}{p_i^*}\langle|M_{fi}|^2\rangle \quad (\ p_i^* = p_f^* = E \text{ in the C.o.M }\ )$$

$$= \frac{1}{4} \times \frac{1}{64\pi^2 s}\left[2\left(-\frac{e^2}{t}2s\right)^2 + 2\left(-\frac{e^2}{t}2s\right)^2 \cos^2\frac{\theta}{2}\right]$$

$$= \frac{2e^4 s^2}{t^2} \times \frac{1}{64\pi^2 s}\left(1+\cos^2\frac{\theta}{2}\right)^{1}$$

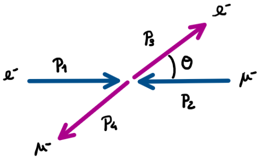
$$\frac{d\sigma}{d\Omega} = \frac{2\alpha^2}{s} \cdot \frac{1+\frac{1}{4}(1+\cos\theta)^2}{(1-\cos\theta)^2}$$

[1] where we use $t = -s\sin^2\frac{\theta}{2} = -s \times (1-\cos\theta)/2$ in the c.o.m. frame and $\alpha = e^2/4\pi$.

Lorentz-invariant form

- The matrix element is Lorentz invariant, but since it is currently expressed using the scattering angle in the C.o.M. frame, we seek a frame-independent form:

$$\langle |M_{fi}|^2 \rangle = \frac{1}{4}\left(|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2 + |M_{LL}|^2\right)$$
$$= \frac{2e^4 s^2}{t^2}\left(1 + \cos^2\frac{\theta}{2}\right)$$



- In the C.o.M. frame $(m_e \sim m_\mu \sim 0)$:

$$\left.\begin{array}{l} p_1 = (E,0,0,E); \quad p_2 = (E,0,0,-E) \\ p_3 = (E, E\sin\theta, 0, E\cos\theta); \\ p_4 = (E, -E\sin\theta, 0, -E\cos\theta) \end{array}\right\} \begin{array}{l} s = (p_1+p_2)^2 = 2p_1\cdot p_2 = 4E^2 \\ t = (p_1-p_3)^2 = -2p_1\cdot p_3 = -4E^2(1-\cos\theta) \\ u = (p_1-p_4)^2 = -2p_1\cdot p_4 = -4E^2(1+\cos\theta) \end{array}$$

*Valid in any frame

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1\cdot p_2)^2 + (p_1\cdot p_4)^2}{(p_1\cdot p_3)^2} \equiv 2e^4\left(\frac{s^2+u^2}{t^2}\right)$$

The End

References

- Thompson, M. (2013). *Modern Particle Physics*. Cambridge University Press.
- Gonçalo, R. (2025). *High Energy Physics; QED: quantum electrodynamics*, Lecture Slides.