Berry Curvature and Spin-One Color Superconductivity

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In a collaboration with Yi Yin (The Chinese University of Hong Kong)

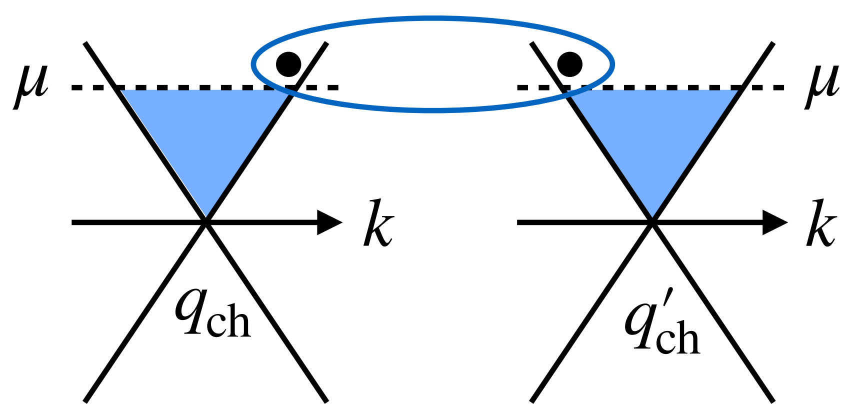
Aim of the talk

	Chirality of fermions	Consequences
Normal fluids		• Chiral magnetic effect • Chiral vortical effect • …
Superconductors/ Superfluids		This talk

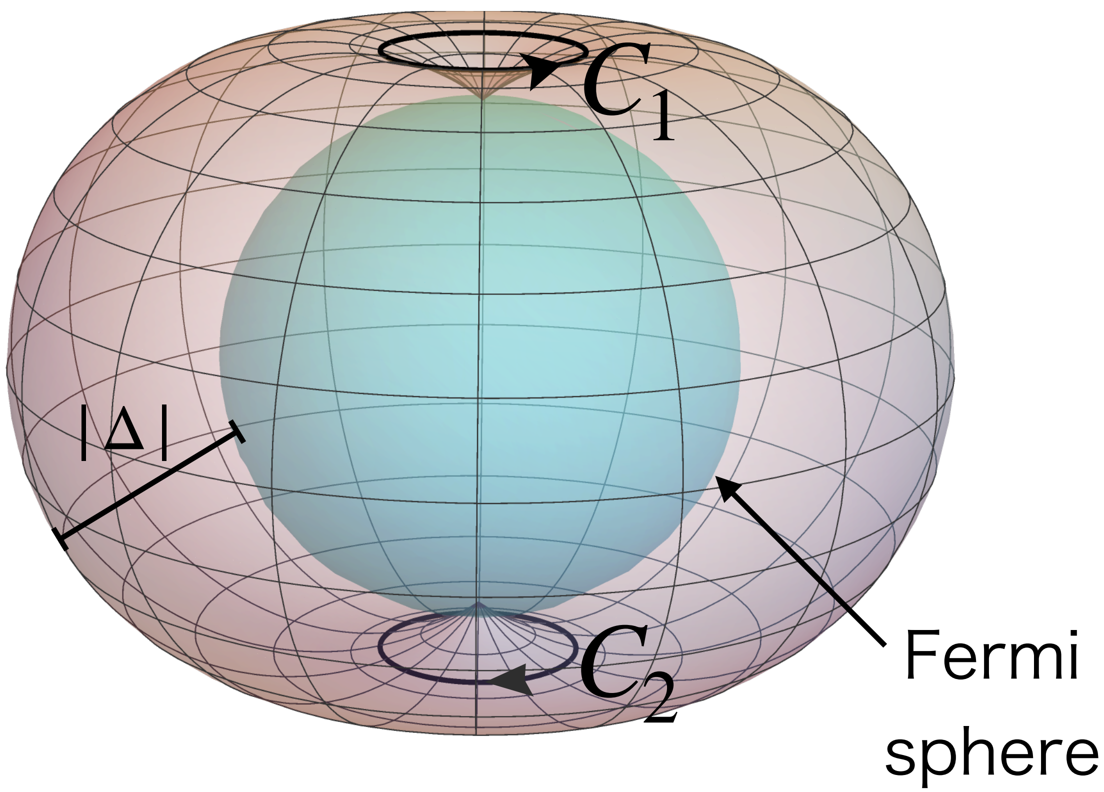
Topological nodal Cooper pairing

Li and Haldane (2018)
Murakami and Nagaosa (2003)

The pairing gap function: $\Delta = |\Delta| e^{i\theta}$



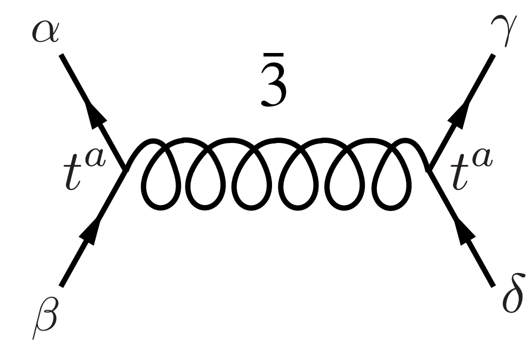
$q_{ch}, q'_{ch} = \pm 1/2$



$$g \sim \sum_N \oint_{C_N} \mathrm{d}\boldsymbol{t} \cdot \nabla_{\boldsymbol{k}} \theta = 2\Delta q_{ch}$$

The pairing monopole charges: $\Delta q_{ch} \equiv q_{ch} - q'_{ch} \neq 0 \Rightarrow$ nodes with $g \neq 0$

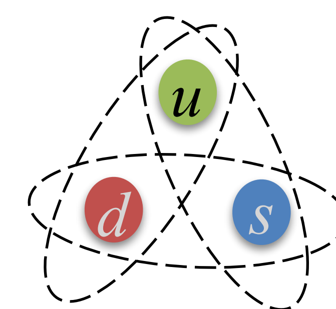
Color Superconductivity (CSC)



- QCD at high baryon density $(\mu_{\mathrm{B}} \gg \Lambda_{\mathrm{QCD}}, T)$

  Attractive one-gluon exchange interaction

  ⟶ BCS mechanism



- Three light flavors: Spin-0 (Color flavor locking phase)
- Finite strange quark mass

  ⟶ Fermi momentum mismatch

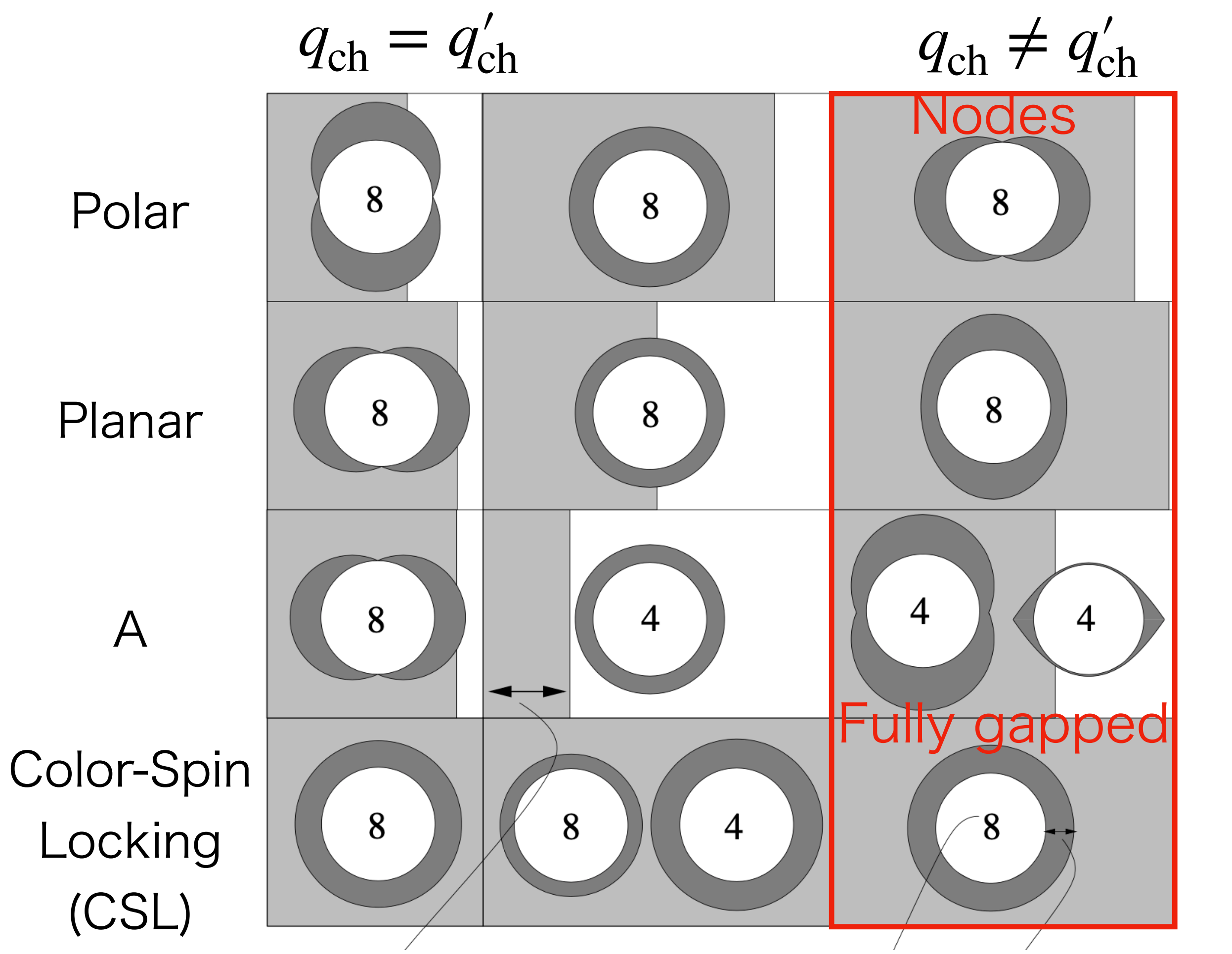
  ⟶ Spin-1 single flavor pairings (uu, dd, ss) or (ss with ud (2SC))

  - Color antisymmetric (attractive channel)
  - Spin symmetric (Pauli principle)

    ✓ No flavor degrees of freedom

    ✓ Spin-1 CSC

Puzzles in spin-1 CSC



Schäfer (2000), Table from Schmitt (2005)

$$g = 2\Delta q_{ch} \equiv 2(q_{ch} - q'_{ch})$$

Generic nodes for $q_{ch} \neq q'_{ch}$

Li and Haldane (2018)

Murakami and Nagaosa (2003)

What are missing?

NS and Yi (2025)

- Conventional work: Limited to spin (chirality) contribution
- We incorporate additional quantum numbers (color) of the pairings:

$$g = 2(\Delta q_{\mathrm{ch}} + \Delta q_{\mathrm{c}})$$

Consequences:

1. Δq_{ch} can be cancelled by the color contribution Δq_{c} yielding $g = 0$

$\longrightarrow$ Explanation for the fully gaped phases

2. $\Delta q_{\mathrm{ch}} \neq 0$ manifests as gapless modes that carry unconventional

Berry monopoles charges $\pm 3/2 \neq$ ($\pm 1/2$ of a Weyl fermion)

QCD with one flavor

- Mean-field (BdG) Hamiltonian:

$$H = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \left(\psi^\dagger_{\mathrm{R}}, \psi^\dagger_{\mathrm{L},\mathcal{C}}\right) \begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{k} - \mu & M \\ M^\dagger & \boldsymbol{\sigma}\cdot\boldsymbol{k} + \mu \end{pmatrix} \begin{pmatrix} \psi_{\mathrm{R}} \\ \psi_{\mathrm{L},\mathcal{C}} \end{pmatrix}$$

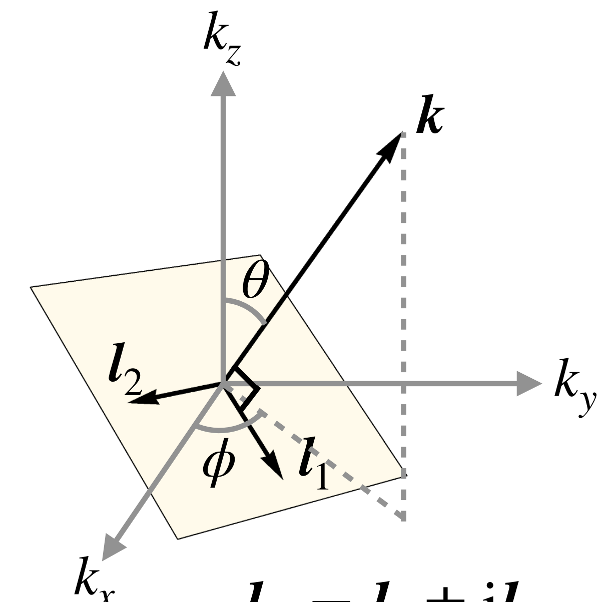
  - Gap matrix: $M = (P_+ \sigma_i^\perp)\Delta_{ia} J_a = (\mathrm{spin}) \otimes (\mathrm{color})$

$$P_+ = (1 + \boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}})/2\,, \quad \sigma_i^\perp = (\delta_{ij} - \hat{k}_i \hat{k}_j)\sigma_j\,, \quad (J_a)_{bc} = -\,\mathrm{i}\epsilon_{abc}$$

  - Order parameter (with overall gap function Δ_0):

$$\Delta_{ia} = \begin{cases} \Delta_0 \delta_{i3}\delta_{a3} & (\text{Polar}) \\ \Delta_0 \delta_{ia} & (\text{CSL}) \end{cases}$$

QCD with one flavor



$$\boldsymbol{l}_\pm = \boldsymbol{l}_1 \pm \mathrm{i}\boldsymbol{l}_2$$

Non-Abelian Berry connection

Wilczek and Zee (1984)

- $\boldsymbol{A}_{\lambda,mn} = -\mathrm{i}\phi^\dagger_{\lambda,m}\nabla_{\boldsymbol{k}}\phi_{\lambda,n}/N_\lambda$ for gapped/gapless modes, likewise $\boldsymbol{A}'_{\lambda,mn}$

- $\phi, \phi' \sim (\text{spin}) \otimes (\text{color}) \longrightarrow \boldsymbol{A}_{\lambda,mn} = (\boldsymbol{A}_{\mathrm{c},mn} + \delta_{mn}\boldsymbol{A}_{\mathrm{R}})/N_\lambda$ and $\boldsymbol{A}'_{\lambda,mn}$

  Spin: $\boldsymbol{A}_{\mathrm{R}}, \boldsymbol{A}_{\mathrm{L}}$ (Weyl fermions); Color: $\overrightarrow{A}_{\mathrm{c},mn} = -\mathrm{i}\boldsymbol{c}^\dagger_m \overrightarrow{\nabla}_{\boldsymbol{k}}\boldsymbol{c}_n$

- Berry curvature and monopole charges:

$$q = \frac{1}{4\pi}\int_{\mathrm{FS}} \mathrm{d}\boldsymbol{S}\cdot\mathrm{tr}\,\boldsymbol{B} = \frac{1}{4\pi}\int_{\mathrm{FS}} \mathrm{d}\boldsymbol{S}\cdot(\nabla_{\boldsymbol{k}}\times\mathrm{tr}\,\boldsymbol{A}) = q_{\mathrm{ch}} + q_{\mathrm{c}} \text{ and } q' = q'_{\mathrm{ch}} + q'_{\mathrm{c}}$$

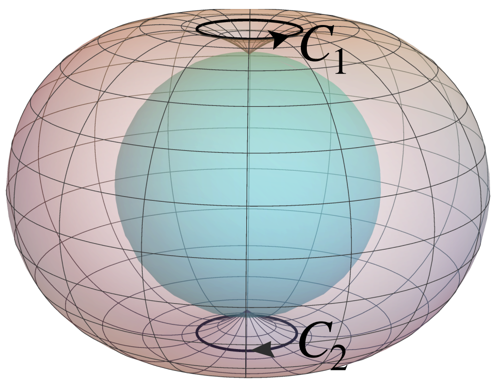
Generalized circulation formula

1. Projected gap function: $\tilde{M}^{\dagger}_{\lambda,mn} \equiv {\phi'_{\lambda,m}}^{\dagger} M^{\dagger} \phi_{\lambda,n}$
2. Gauge invariant "momentum space" velocity field:

$$\boldsymbol{u} \equiv \nabla_{\boldsymbol{k}} \alpha - \mathrm{tr}\,(\boldsymbol{A} - \boldsymbol{A}'), \quad \alpha = -\,\mathrm{i}(\log \det \tilde{M}^{\dagger})/N_{\lambda} \sim (\text{phase of the gap})$$

3. Circulation around nodes where $\tilde{M}$ (and λ) vanish:

$$g \equiv \frac{1}{2\pi} \sum_{N} \oint_{C_N} \mathrm{d}\boldsymbol{t} \cdot \boldsymbol{u}$$

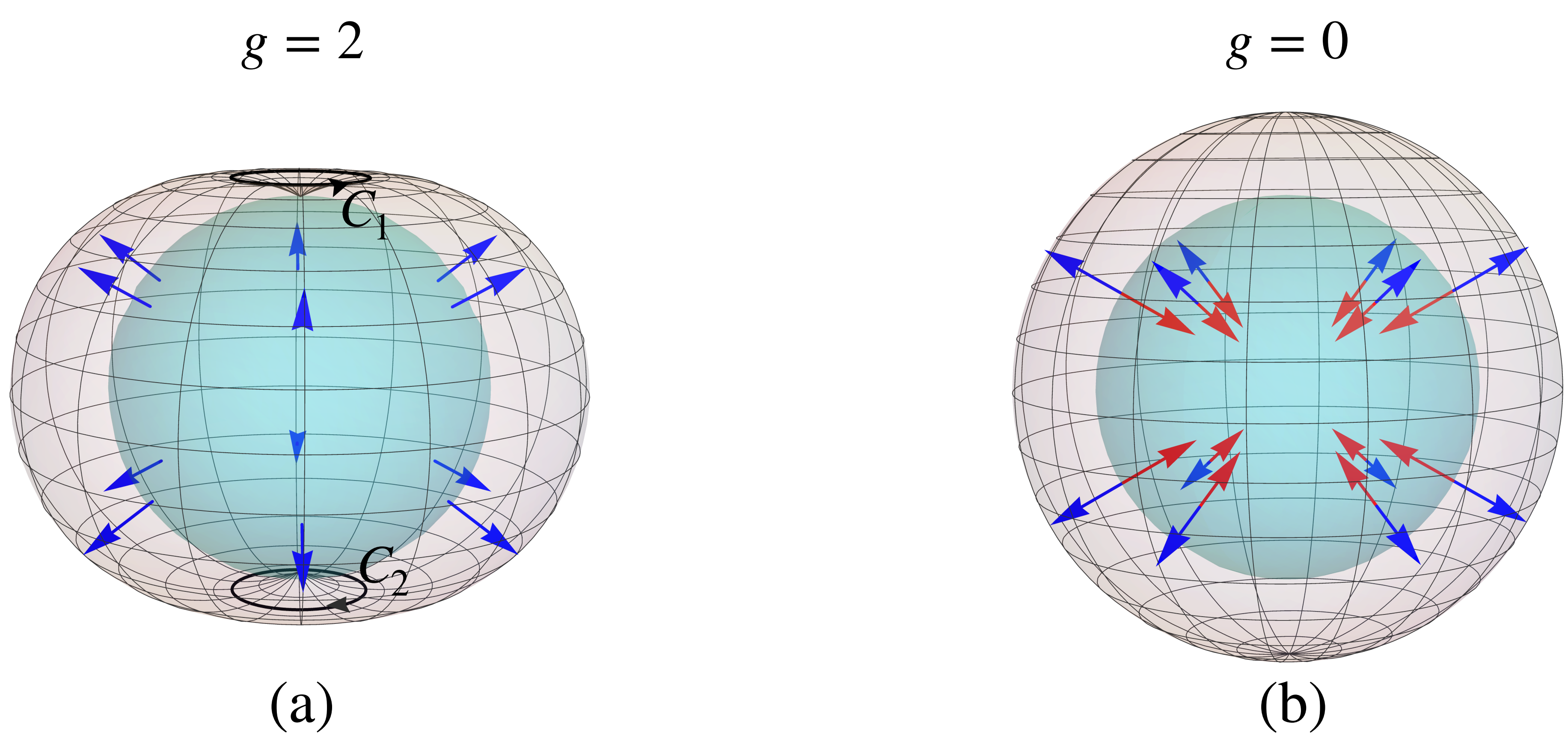


Reverse the loop and use Stokes' law:

$$g = \frac{-1}{2\pi} \iint_{\mathrm{FS}} \mathrm{d}\boldsymbol{S} \cdot (\nabla_{\boldsymbol{k}} \times \boldsymbol{u}) = 2\Delta q = 2(q - q') = 2(\Delta q_{\mathrm{ch}} + \Delta q_{\mathrm{c}}) \longleftarrow \text{Chirality + Color}$$

Abelian case: Li and Haldane (2018) Murakami and Nagaosa (2003)

Polar vs CSL: Circulation



Berry flux by chirality and color helicity

Chirality—Color Cancellation ⟶ Fully Gapped CSL

The sum rule

- Gapless modes appear exactly when nodes close
- Completeness of the color wave function

$\longrightarrow \quad N_\lambda q_{\rm c} + q_{\rm gapless,c} = 0 \quad (N_\lambda = 2)$

Combining with the circulation formula $g = 2(\Delta q_{\rm c} + \Delta q_{\rm ch})$,

$$2\Delta q_{\rm ch} = g + \Delta q_{\rm gapless,c}$$

- Manifestation of $\Delta q_{\rm ch} \neq 0$:

Scenario A (Conventional)

$$2\Delta q_{\rm ch} = g$$

Li and Haldane (2018)

Murakami and Nagaosa (2003)

Scenario B (Novel)

$$2\Delta q_{\rm ch} = \Delta q_{\rm gapless,c}\,, \quad g = 0$$

✓ No nodes

✓ Topological gapless excitations

Gapless excitations

- $(\phi,0)$ and $(0,\phi')$; $\phi_\lambda(\hat{\boldsymbol{k}}) = \boldsymbol{c}_{\rm gapless}(\hat{\boldsymbol{k}}) \otimes \boldsymbol{\xi}_{\rm R}(\hat{\boldsymbol{k}})\,, \quad \phi'_\lambda(\hat{\boldsymbol{k}}) = \boldsymbol{c}'_{\rm gapless}(\hat{\boldsymbol{k}}) \otimes \boldsymbol{\xi}_{\rm L}(\hat{\boldsymbol{k}})$
- "Color" Berry monopole charges:

  Polar: $\boldsymbol{c}_{\rm gapless} = \boldsymbol{c}'_{\rm gapless} = \boldsymbol{e}_{\rm g} \longrightarrow q_{\rm gapless,c} = q'_{\rm gapless,c} = 0$

  CSL: $\boldsymbol{c}_{\rm gapless} \propto \boldsymbol{l}_+, \quad \boldsymbol{c}'_{\rm gapless} \propto \boldsymbol{l}_- \longrightarrow q_{\rm gapless,c} = 1\,, \quad q'_{\rm gapless,c} = -1$

  ($\Delta q_{\rm gapless,c} = 2\Delta q_{\rm ch} = 2$ as expected)

- The total monopole charges in the CSL phase:

$$q_{\rm gapless} = q_{\rm gapless,c} + q_{\rm ch} = \frac{3}{2}\,, \quad q'_0 = -\frac{3}{2}\,,$$

$q_{\rm gapless}\,,\; q'_{\rm gapless}$ = "color helicity" (± 1) + spin helicity $(\pm 1/2)$

c.f., spin helicity only in the polar phase: unpaired quarks

Summary

NS and Yi Yin, arXiv:2411.08005
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- Interplay between Berry curvature and spin-1 color superconductivity
- Generalized formula:

$$g = 2(\Delta q_{\rm ch} + \Delta q_{\rm c}) \quad \Leftrightarrow \quad 2\Delta q_{\rm ch} = g + \Delta q_{\rm gapless,c}$$

Scenario A (Conventional)

$$2\Delta q_{\rm ch} = g$$

Li and Haldane (2018)
Murakami and Nagaosa (2003)

Scenario B (Novel)

$$2\Delta q_{\rm ch} = \Delta q_{\rm gapless,c}\,, \quad g = 0$$

✓ No nodes
✓ Topological gapless excitations

- Potential relevance in condensed matter systems with internal degrees of freedom