Out-of-equilibrium Chiral Magnetic Effect from conventional lattice QCD simulations

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CME from first-principle lattice QCD simulations?

- First lattice measurements in ArXiv:0907.0494 - qualitative demonstration of current correlations with topological charge fluctuations
- Not something we can plug into hydrodynamical equations ...
- How to measure CME in first-principle lattice QCD simulations?

What is CME?

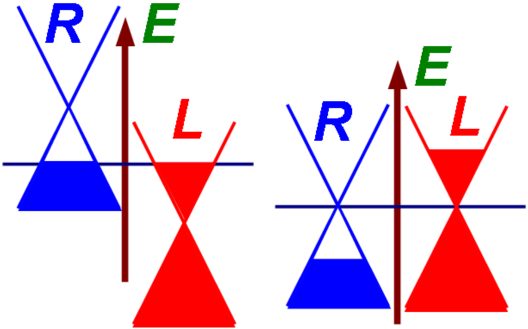
- Paradigmatic CME formula

$$\vec{j} = \frac{\mu_5}{2\pi^2}\,\vec{B}, \tag{1}$$

- $\vec{j}$, $\vec{B}$ and π are all well-defined quantities: we know how to measure/control them
- "Chiral chemical potential" μ_5: additional terms in the Hamiltonian $\hat{\mathcal{H}} \to \hat{\mathcal{H}} + \mu_5 \hat{\mathcal{Q}}_5$
- Introduced as an effective description of chirally imbalanced matter
- Can be modelled/estimated, but certainly absent in $\hat{\mathcal{H}}_{QCD}$

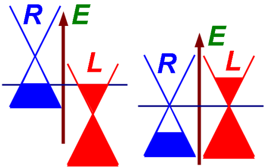
What is μ_5?

- $\hat{\mathcal{Q}}_5$ does not commute with $\hat{\mathcal{H}}$ $\Rightarrow$ Not quite the usual chemical potential
- Unlike the baryonic μ: μ_5 changes the energy spectrum $\epsilon\left(\vec{k}\right)$ of Dirac fermions
- We can consider the thermal equilibrium state



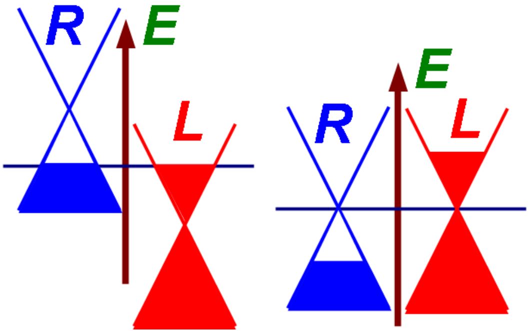
Equilibrium CME with μ_5?

- A lot of enthusiasm initially
- Nonzero results in thermal equilibrium with $\mu_5 \neq 0$ [Yamamoto, PRL107(2011)031601]
- BUT: Non-conserved vector current
- Bloch theorem: no persistent electric currents in thermal equilibrium
- No CME in equilibrium with nonzero μ_5 (properly regularized vector current)
- Recent high-precision results: [Brandt, Endrődi, Garnacho-Velasco, Markó, 2405.09484]



Non-equilibrium CME

- Circumvent Bloch's theorem by making $\vec{B}$ time-dependent? $\Rightarrow$ Does not clarify the meaning of μ_5.
- Chiral imbalance is always induced by non-equilibrium processes
- E.g. sphaleron transitions, external electric field $\parallel \vec{B}$
- States with chirality imbalance are excited states vs thermal equilibrium



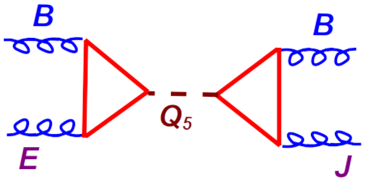
Measuring/simulating non-equilibrium CME

- Simulate the non-equilibrium processes leading to nonzero $\hat{\mathcal{Q}}_5$
  - Sphaleron transitions
  - Parallel electric and magnetic fields
- Full out-of-equilibrium simulations (classical gauge fields + quantum fermions for QCD)
- First-principles: linear response approximation
  - Small electric fields
  - Small fluctuations of $\hat{\mathcal{Q}}_5$

$$\frac{d\langle \hat{\mathcal{Q}}_5 \rangle}{dt} = \frac{\vec{E}\cdot\vec{B}}{2\pi^2} + \frac{\langle \vec{\mathcal{E}}_a \cdot \vec{\mathcal{B}}_a \rangle}{2\pi^2} + \\ +2im_f \int d^3\vec{x} \langle \hat{q}^\dagger \gamma_5 \gamma_0 \hat{q} \rangle. \tag{2}$$

Small electric fields: negative magnetoresistance

- Quadratic dependence of electric conductivity on $\vec{B}$
- Disadvantage: CME/axial anomaly is not the only process contributing to $\vec{B}$ dependence of electric conductivity



Axial charge from thermal fluctuations

- Axial charge exhibits thermal fluctuations even without electric/magnetic fields: $\langle\, \hat{\mathcal{Q}}_5 \,\rangle = 0$ but $\langle\, \hat{\mathcal{Q}}_5^2 \,\rangle \neq 0$
- Chirally imbalanced states are thermally populated anyway
- Lead to thermal fluctuations of CME electric current
- **Let's capture these correlations!**

Axial charge - electric current correlator

- In Euclidean/imaginary time:

$$G_{5z}(\tau) = \mathcal{Z}^{-1} \mathrm{Tr}\left(\hat{\mathcal{J}}_z e^{-\tau \hat{\mathcal{H}}} \hat{\mathcal{Q}}_5 e^{-(\beta-\tau)\hat{\mathcal{H}}}\right) \quad (3)$$

- Retarded correlator in real time:

$$G^R_{5z}(t) = i\,\theta(t)\,\langle \left[\hat{\mathcal{Q}}_5(0), \hat{\mathcal{J}}_z(t)\right] \rangle, \quad \hat{\mathcal{J}}_z(t) = e^{i\hat{\mathcal{H}}t} \hat{\mathcal{J}}_z e^{-i\hat{\mathcal{H}}t}, \quad (4)$$

- Analytic continuation relations:

$$G_{5z}(w_k) = -G^R_{5z}(w \to i w_k) \quad (5)$$

- $w_k = 2\pi k\, T$ - Matsubara frequencies

Real-time interpretation

- Assume an infinitesimal time-dependent perturbation of the Hamiltonian: $\hat{\mathcal{H}}(t) = \hat{\mathcal{H}}_0 + \mu_5(t)\,\hat{\mathcal{Q}}_5$
- How does $\langle \hat{\mathcal{J}}_z \rangle (t)$ depend to time?

$$\langle \hat{\mathcal{J}}_z(t) \rangle = \int_{-\infty}^{t} dt'\, G^R_{5z}(t - t')\, \mu_5(t'), \tag{6}$$

- $\mu_5(t)$ is now a time-dependent perturbation.

Non-interacting fermions

- Dirac Hamiltonian with background magnetic field

$$H = \begin{pmatrix} -i\sigma_k \nabla_k & m \\ m & i\sigma_k \nabla_k \end{pmatrix} \tag{7}$$

- Uniform magnetic field $\vec{B} = \{0, 0, B_z\}$

Unregularized result: technical details

$$G^{0}_{5z}(\tau) = \frac{N_B}{V}\sum_{k_z}\sum_{s=\pm 1}\frac{k_z^2}{E^2}\frac{e^{-\beta\, s\, E}}{\left(1+e^{-\beta\, s\, E}\right)^2} + $$
$$+ \frac{N_B}{V}\sum_{k_z}\sum_{s=\pm 1}\frac{m^2}{E^2}\frac{e^{(\beta/2-\tau)\,2\,s\,E}}{\left(1+e^{-\beta\, s\, E}\right)\left(1+e^{+\beta\, s\, E}\right)},$$
$$E = \sqrt{k_z^2 + m^2} \qquad (8)$$

- Only the lowest Landau level contributes
- The first term is τ-independent
- The second term is UV-finite and vanishes as $m \to 0$

Chiral limit and regularization

- Unregularized result at $m \to 0$

$$G^0_{5z}(\tau)\Big|_{m=0} = \frac{B_z}{8\pi^2} \int\limits_{-\infty}^{+\infty} \frac{dk_z}{\cosh^2(\beta k_z/2)} = \frac{B_z\, T}{2\pi^2}. \tag{9}$$

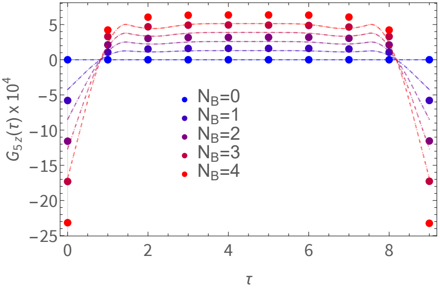
- Pauli-Villars regularization: subtract unregularized result at $m \to +\infty$

$$G^0_{5z}(\tau)\Big|_{m\to+\infty} = \frac{B_z}{2\pi^2}\,\delta(\tau). \tag{10}$$

- Final result for Euclidean-time correlator:

$$\begin{aligned} G_{5z}(\tau)|_{m=0} &= G^0_{5z}(\tau)\Big|_{m=0} - G^0_{5z}(\tau)\Big|_{m\to+\infty} = \\ &= \frac{B_z\, T}{2\pi^2} - \frac{B_z\,\delta(\tau)}{2\pi^2}. \end{aligned} \tag{11}$$

Lattice results



- All observables in $G_{5z}(\tau)$ are well-defined and controlled
- $\Rightarrow$ Straightforward to measure on the lattice
- $SU(2)$ lattice gauge theory
- $N_f = 2$ light dynamical quarks
- $30^3 \times 10$ lattice, magnetic fluxes $N_B = 0 \ldots 4$.

Equilibrium state with μ_5

- $\hat{\mathcal{H}} \to \hat{\mathcal{H}} + \mu_5 \hat{\mathcal{Q}}_5$
- Expand to first order in μ_5
- Thermal expectation values:

$$\langle \, \hat{\mathcal{Q}}_5 \, \rangle = \mu_5 \int\limits_0^\beta G_{5z}\left(\tau\right) = 0 \tag{12}$$

- This zero was measured in [Brandt, Endrődi, Garnacho-Velasco, Markó, 2405.09484]

Real-time response

- Unregularized result for the retarded propagator:

$$G^{R0}_{5z}(w) = 0, \quad m \to 0 \tag{13}$$

- Pauli-Villars regulator:

$$G^{R0}_{5z}(w) = -\frac{B_z}{2\pi^2}, \quad m \to +\infty \tag{14}$$

- Final result - note how the interpretation of μ_5 is slightly changed:

$$G^{R}_{5z}(w) = \frac{B_z}{2\pi^2} \Rightarrow \langle \hat{\mathcal{J}}_z \rangle (t) = \frac{B_z}{2\pi^2} \mu_5(t) \tag{15}$$

- Compare with Fourier transform of the Euclidean correlator:

$$G_{5z}(w_k) = -\frac{B_z}{2\pi^2}(1 - \delta_{k,0}) \tag{16}$$

Relation to real-time anomaly

- Add a small time-dependent $U(1)$ gauge field to the Hamiltonian: $\hat{\mathcal{H}} \to \hat{\mathcal{H}} + A_z(t)\,\hat{\mathcal{J}}_z$.
- What time dependence of $\langle Q \rangle(t)$ does this perturbation induce?
- Real-time anomaly equation:

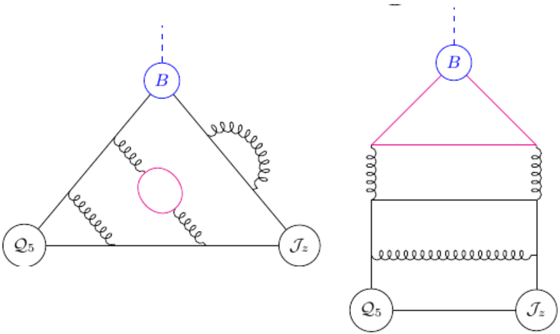
$$\frac{d\langle \hat{\mathcal{Q}}_5 \rangle}{dt} = \frac{\dot{A}_z\, B_z}{2\pi^2} \Rightarrow \langle \hat{\mathcal{Q}}_5 \rangle(t) = A_z(t)\,\frac{B_z}{2\pi^2} \tag{17}$$

- In linear response approximation, retarded correlator that is complex conjugate to G^R_{5z}:

$$\bar{G}^R_{z5}(t) = i\,\theta(t)\,\langle \left[\hat{\mathcal{J}}_z(0), \hat{\mathcal{Q}}_5(t)\right] \rangle \tag{18}$$

- Corrections to $G^R_{5z}(t)$ are strongly constrained

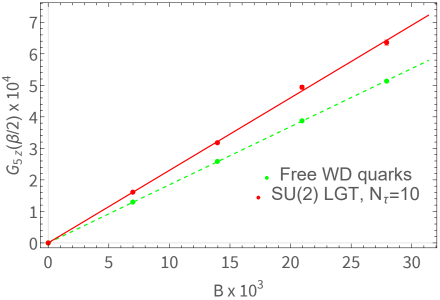
Possible corrections?



- No corrections from dressing the triangle
- Corrections from disconnected fermion diagrams are still possible

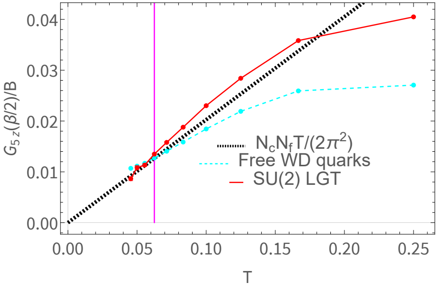
- Retarded correlator $G^R_{5z}(w)$ is real-valued
- No pole singularities at finite frequencies w
- $\Rightarrow S_{5z}(w) = \frac{1}{\pi}\mathrm{Im} G^R_{5z}(w) = 0$
- Time-dependent $\mu_5(t)$ does not perform work
- Stark contrast with the usual AC conductivity

Some more lattice results



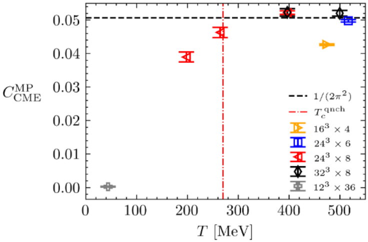
- Plateau height $G_{5z}\,(\beta/2)$ as a function of magnetic field strength
- Solid line is a linear fit

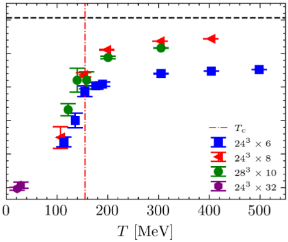
Some more lattice results



- The slope of the linear dependence of $G_{5z}(\beta/2)$ on the magnetic field B as a function of temperature (in lattice units)
- Free-fermion continuum result $G_{5z}(\beta/2)/B = N_c N_f T/(2\pi^2)$
- Same for free Wilson-Dirac fermions
- No noticeable change at the crossover line (magenta line)!

Some more lattice results





- In [Brandt, Endrődi, Garnacho-Velasco, Markó, Valois, arXiv:2502.01155], a suppression of CME is reported at low temperatures
- For our data, interacting lattice results also fall below non-interacting ones

Conclusions

- New lattice setup for measuring the Chiral Magnetic Effect
- Measurements can be made on finite-T gauge ensembles with nonzero magnetic field
- All lattice observables are well-defined
- Frequency dependence can also be explored (modulo the usual analytic continuation difficulties)