Wigner-Weyl calculus for non-Abelian gauge theory

Talk at the international workshop

Weyl and Dirac Semimetals as a Laboratory for High-Energy Physics,

Braga, Portugal, 25 June 2025

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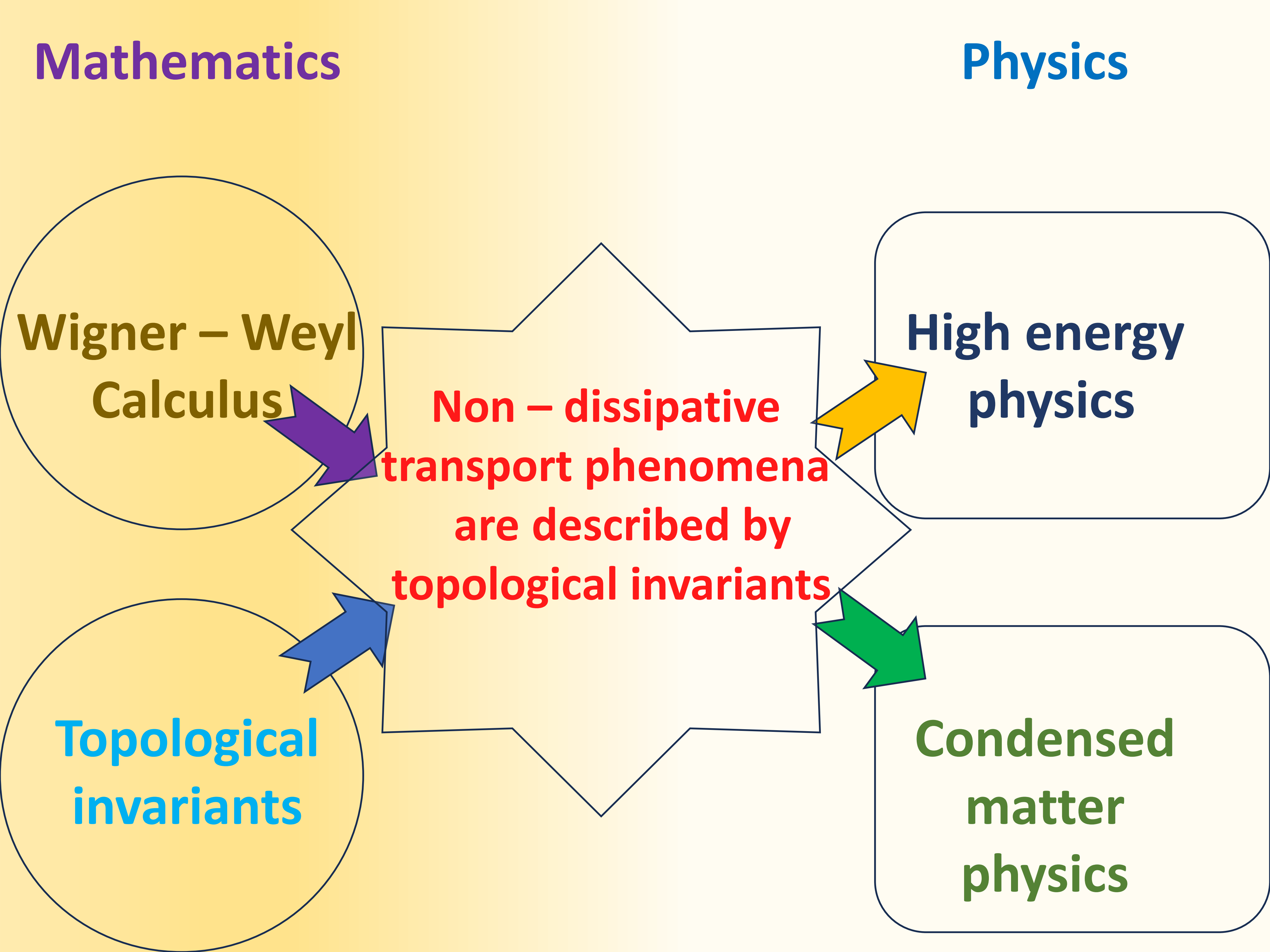
work performed in collaboration with D.P.Xavier

Ariel University Israel

Based of the preprints

Chiral anomaly in inhomogeneous systems
with nontrivial momentum space topology
PD Xavier, MA Zubkov
arXiv preprint arXiv:2506.05066

Generalized Wigner-Weyl calculus for
gauge theory and non-dissipative transport
PD Xavier, MA Zubkov
arXiv preprint arXiv:2410.06952


Mathematics
Physics
Wigner – Weyl Calculus
Topological invariants
Non – dissipative transport phenomena are described by topological invariants
High energy physics
Condensed matter physics

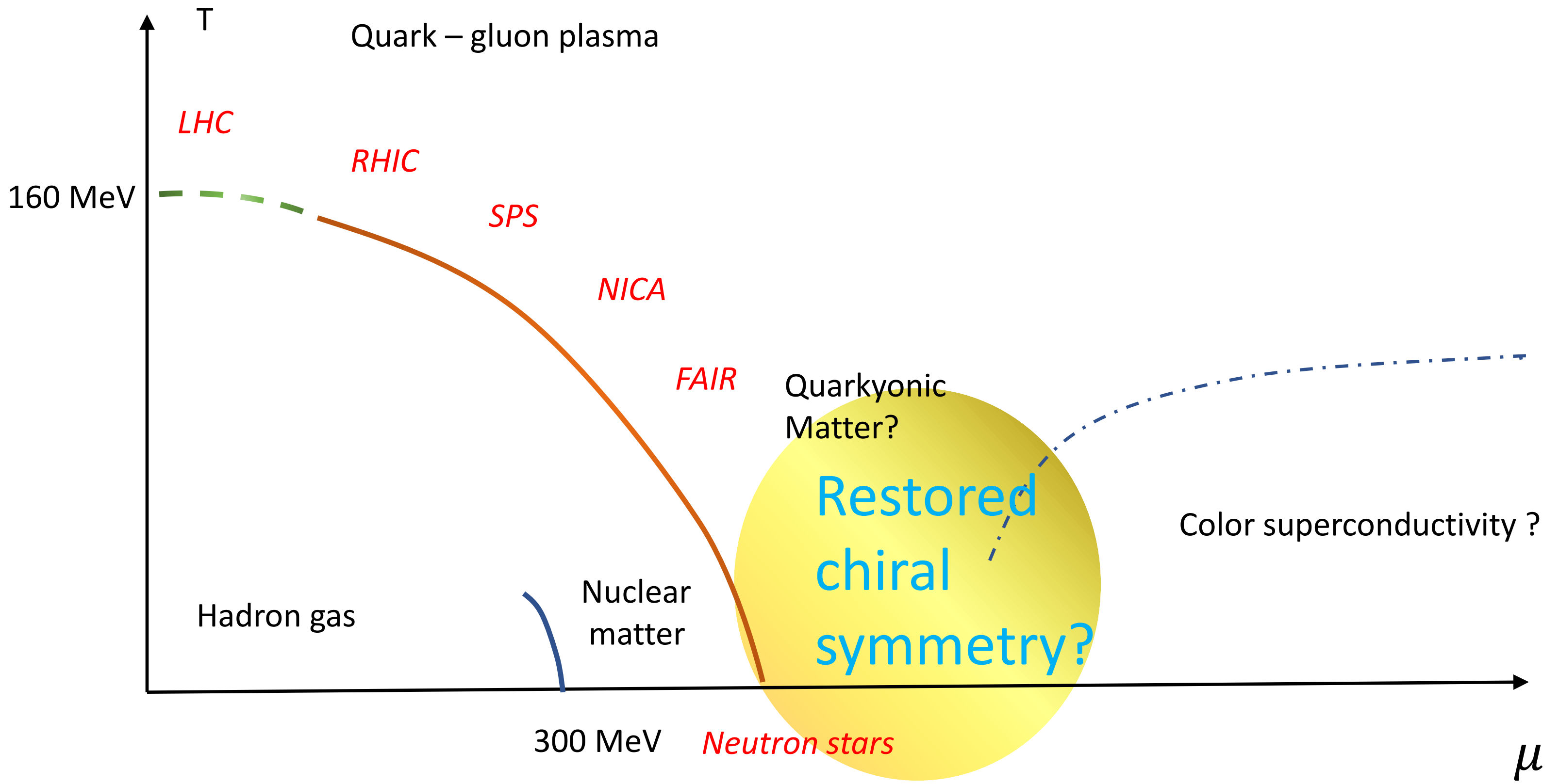
Non – dissipative transport in quark matter

Chiral separation effect (CSE): Axial current in the presence of magnetic field

Chiral vortical effect (CVE): Axial current in the presence of rotation

Chiral magnetic effect (CME): Vector current in the presence of magnetic field

And chiral disbalance



Non – dissipative transport in condensed matter

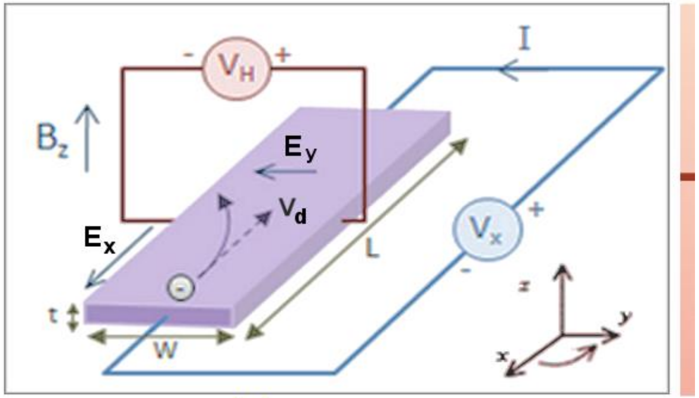
Quantum Hall effect (QHE): Electric current orthogonal to electric field

Chiral separation effect (CSE): Axial current in the presence of magnetic field

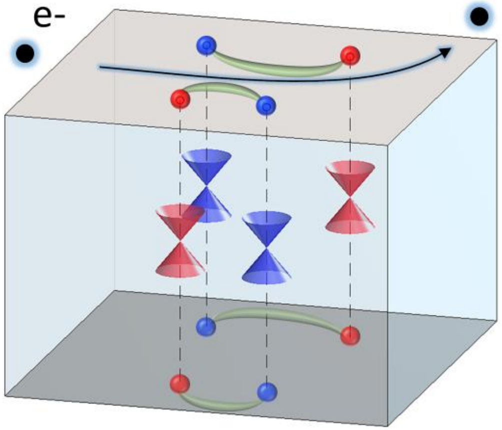
Chiral vortical effect (CVE): Axial current in the presence of rotation

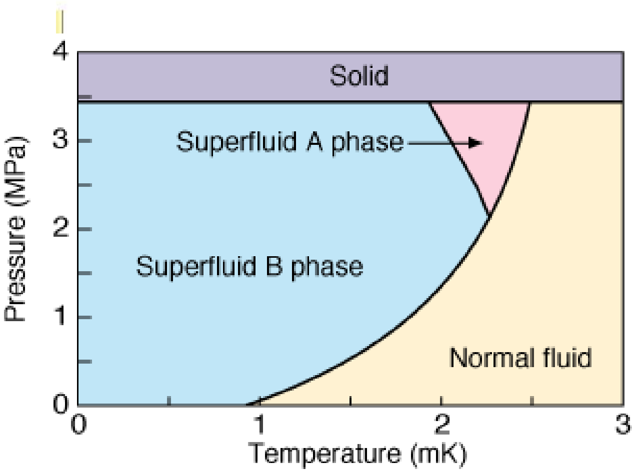
Chiral magnetic effect (CME): Vector current in the presence of magnetic field

And chiral disbalance



(a) Hall Effect





2d materials: QHE 3d Weyl semimetals: CSE, CME, QHE He3-A superfluid:CVE

We are going to extend the consideration to the non – Abelian versions of the chiral separation effect and quantum Hall effect.

We also would like to obtain expression for chiral anomaly in the presence of external non - Abelian gauge field in the case when topology of fermions in momentum space is nontrivial.

(To the best of our knowledge this has not been done in the past)

Conventional Wigner – Weyl calculus

model with fermions

Covariant Wigner – Weyl calculus

model with fermions

$$Z = \int D\bar{\psi} D\psi \; e^{S[\psi,\bar{\psi}]}$$

typical action

typical action

$$S[\bar{\psi},\psi] = \int d^4x \bar{\psi}(x)\hat{Q}(\partial_x)\psi(x)$$

$$\hat{Q}(\partial_x) = i\gamma^\mu \partial_\mu - M$$

$$Q = \sum_{|\alpha| \le m} c_\alpha(x)(-iD)^\alpha$$

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4),\; |\alpha| := \sum_\mu \alpha_\mu$$

$$(-i\partial)^\alpha := \prod_\mu (-i\partial_\mu)^{\alpha_\mu}$$

Green function

Green function

$$\hat{G} := \hat{Q}^{-1}$$

Euclidean space - time

conventional Wigner – Weyl calculus

Weyl symbol of operator

$$A_W(x,p) \equiv \int_{-\infty}^{\infty} dy e^{-ipy} \langle x + \frac{y}{2} | \hat{A} | x - \frac{y}{2} \rangle = \int_{-\infty}^{\infty} dq e^{iqx} \langle p + \frac{q}{2} | \hat{A} | p - \frac{q}{2} \rangle$$

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covariant Wigner – Weyl calculus

Weyl symbol of operator

$$X_W(x,p) := \int d^4y\, e^{ipy}\, U(x, x - y/2) \langle x - y/2 | \hat{X} | x + y/2 \rangle U(x + y/2, x)$$

$$U(y,x) = \mathrm{Pexp}\left(i \int_{x \to y} dz^\mu A_\mu(z) \right)$$

$$X_W(x,p) = \int d^4y\, e^{ipy} \langle x | e^{-\frac{i}{2} y \hat{\pi}} \hat{X} e^{-\frac{i}{2} y \hat{\pi}} | x \rangle$$

where $\hat{\pi}_\mu := \hat{p}_\mu - A_\mu(\hat{x})$

conventional Wigner – Weyl calculus

Moyal product

$$(f \star g)(x,p) := (2\pi)^{-8} \int d^4y d^4k d^4y' d^4k'\, e^{-iy(k-p)-iy'(k'-p)} f(x-y'/2,k) g(x+y/2,k')$$

the product of two operators

$$\boxed{(AB)_W(x,p) \equiv A_W(x,p) \star B_W(x,p)}$$

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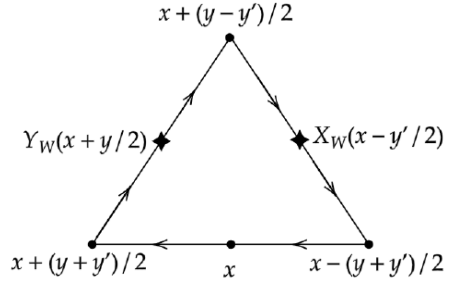
covariant Wigner – Weyl calculus

Star product

$$\boxed{X_W \bigstar Y_W := (\hat{X}\hat{Y})_W}$$

$$\begin{aligned}
&(X_W \bigstar Y_W)(x,p)\\
&=(2\pi)^{-8} \int d^4y d^4k d^4y' d^4k'\, e^{-iy(k-p)-iy'(k'-p)} \times\\
&U(x, x-(y+y')/2)\, U(x-(y+y')/2, x-y'/2)\, X_W(x-y'/2,k)\, U(x-y'/2, x+(y-y')/2)\\
&U(x+(y-y')/2, x+y/2)\, Y_W(x+y/2,k')\, U(x+y/2, x+(y+y')/2)\, U(x+(y+y')/2, x)
\end{aligned}$$

Wilson loop



conventional Wigner – Weyl calculus

Moyal product

$$A_W(x,p) \star B_W(x,p) = A_W(x,p) e^{\overleftrightarrow{\Delta}} B_W(x,p)$$

$$\overleftrightarrow{\Delta} \equiv \frac{i}{2}\left(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x\right)$$

the product of two operators

$$\boxed{(AB)_W(x,p) \equiv A_W(x,p) \star B_W(x,p)}$$

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covariant Wigner – Weyl calculus

Moyal product

$$\boxed{X_W \bigstar Y_W := (\hat{X}\hat{Y})_W}$$

$$X_W(x,p) \bigstar Y_W(x,p) =$$

$$\left(e^{\frac{i}{2}(\overrightarrow{\partial}_{p_1} + \overrightarrow{\partial}_{p_2})\overrightarrow{D}_x} e^{-\frac{i}{2}\overrightarrow{\partial}_{p_1}\overrightarrow{D}_x} X_W(x,p_1) e^{-\frac{i}{2}\overrightarrow{D}_x \overleftarrow{\partial}_{p_1}}\right.$$

$$\left. e^{-\frac{i}{2}\overrightarrow{\partial}_{p_2}\overrightarrow{D}_x} Y_W(x,p_2) e^{-\frac{i}{2}\overleftarrow{\partial}_{p_2}\overrightarrow{D}_x} e^{\frac{i}{2}(\overleftarrow{\partial}_{p_1} + \overleftarrow{\partial}_{p_2})\overrightarrow{D}_x}\right) \times 1 \Bigg|_{p_1=p_2=p}$$

Conventional Wigner – Weyl calculus

model with fermions

Covariant Wigner – Weyl calculus

model with fermions

$$Z = \int D\bar{\psi} D\psi \; e^{S[\psi,\bar{\psi}]}$$

typical action

typical action

$$S[\bar{\psi},\psi] = \int d^4x \bar{\psi}(x)\hat{Q}(\partial_x)\psi(x)$$

$$\hat{Q}(\partial_x) = i\gamma^\mu\partial_\mu - M$$

$$Q = \sum_{|\alpha|\leq m} c_\alpha(x)(-iD)^\alpha$$

$$\alpha = (\alpha_1,\alpha_2,\alpha_3,\alpha_4),\; |\alpha| := \sum_\mu \alpha_\mu$$

$$(-i\partial)^\alpha := \prod_\mu (-i\partial_\mu)^{\alpha_\mu}$$

Green function

Green function

$$\boxed{(\hat{Q}\hat{G})_W = Q_W \star G_W = 1}$$

$$\boxed{G_W(x,p) \bigstar Q(x,p) = 1}$$

Conventional Wigner – Weyl calculus

model with fermions

Covariant Wigner – Weyl calculus

model with fermions

$$Z = \int D\bar{\psi} D\psi \; e^{S[\psi,\bar{\psi}]}$$

typical action

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$$S[\bar{\psi},\psi] = \int d^4x \bar{\psi}(x)\hat{Q}(\partial_x)\psi(x)$$

Green function

$$\boxed{(\hat{Q}\hat{G})_W = Q_W \star G_W = 1}$$

Green function

$$\boxed{G_W(x,p)\bigstar Q(x,p) = 1}$$

$$Q_W(x,p) = \int d^4y \, e^{ipy} \langle x - y/2 | \, \hat{Q}^{(A=0)} \, | x + y/2 \rangle$$

$$Q_W(x,p) \equiv (\hat{Q})_W(x,p) = Q(x,p)$$

$$Q(x,p) = \sum_{|\alpha| \le m} o_\alpha(x) p^\alpha$$

Conventional Wigner – Weyl calculus

Green function

$$(\hat{Q}\hat{G})_W = Q_W \star G_W = 1$$

$$G^{(1)}_{0,W}(x,p) = -\frac{\partial G^{(0)}_{0,W}}{\partial p_\mu}\delta A_\mu - \frac{i}{2} G^{(0)}_{0,W} \star \frac{\partial Q_W}{\partial p_\mu} \star \frac{\partial G^{(0)}_{0,W}}{\partial p_\nu}\delta F_{\mu\nu}$$

Covariant Wigner – Weyl calculus

$$G_W(x,p) \bigstar Q(x,p) = 1$$

$$Q(x,-iD) = \sum_{|\alpha|\le m} o_\alpha(x) \circ (-iD)^\alpha$$

$$o_\alpha(x) \circ (-iD)^\alpha = \frac{1}{2^{|\alpha|}}\{...\{o_\alpha(x),(-iD_1)\}...(-iD_1)\}(-iD_2)\}...(-iD_2)\}...(-iD_4)\}$$

$$Q(x,p) = \sum_{|\alpha|\le m} o_\alpha(x) p^\alpha$$

Conventional Wigner – Weyl calculus

Green function

Covariant Wigner – Weyl calculus

$$(\hat{Q}\hat{G})_W = Q_W \star G_W = 1$$

$$G_W(x,p) \bigstar Q(x,p) = 1$$

$$Q(x,p) = \sum_{|\alpha| \leq m} o_\alpha(x) p^\alpha$$

$$G_W(x,p,z) = \sum_{n \geq 0} G^{(n)}(x,p,z)$$

$G^{(n)}(x,p,z)$ contains n powers of D_z.

$$G^{(2)}(x,p,z) = -\frac{i}{2} G^{(0)}(x,p) \star \partial_{p_\mu} Q(x,p) \star \partial_{p_\nu} G^{(0)}(x,p) F_{\mu\nu}(z)$$

QUANTUM HALL EFFECT

Conventional QHE
(normal Wigner –Weyl calculus)

$$\boxed{(\hat{Q}\hat{G})_W = Q_W \star G_W = 1}$$

Non – Abelian QHE
(Covariant Wigner – Weyl)

$$\boxed{G_W(x,p)\bigstar Q(x,p) = 1}$$

$$G^{(2)}(x,p,z) = -\frac{i}{2}G^{(0)}(x,p) \star \partial_{p_\mu} Q(x,p) \star \partial_{p_\nu} G^{(0)}(x,p) F_{\mu\nu}(z)$$

Abelian Vector current

$$j_k(x) = \frac{\delta \log Z}{\delta A_k(x)}$$

Non – Abelian vector current

$$\langle J_\mu(x)\rangle = -\mathrm{tr}_D \int \frac{d^4p}{(2\pi)^4} G_W \partial_{p_\mu} Q$$

Response to (chromo) Electric field in 2+1 D

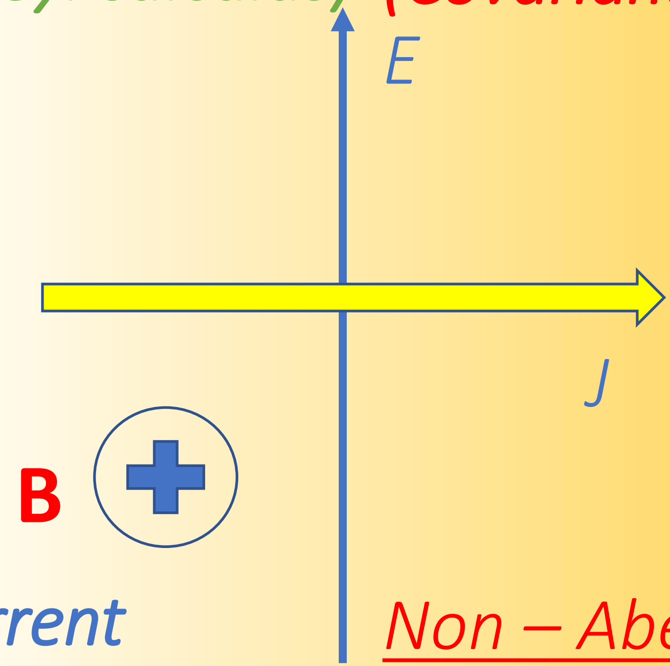
$$\bar{J}_i^{v,QHE} = \frac{1}{2\pi}\epsilon_{ij} M_3 E_j$$

$$\boxed{M_3 = -\frac{1}{S\,24\pi^2}\left[\int d^2x \int \mathrm{tr}_D \left(G^{(0)} \star dQ \star \wedge dG^{(0)} \star \wedge dQ\right)\right]_{reg}}$$

QUANTUM HALL EFFECT

Conventional QHE
(normal Wigner –Weyl calculus)

Non – Abelian QHE
(Covariant Wigner – Weyl)



Abelian Vector current

Non – Abelian vector current

Response to (chromo) Electric field in 2+1 D

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CHIRAL SEPARATION EFFECT

Conventional QHE (normal Wigner –Weyl calculus)

Non – Abelian QHE (Covariant Wigner – Weyl)

$$(\hat{Q}\hat{G})_W = Q_W \star G_W = 1$$

$$G_W(x,p) \bigstar Q(x,p) = 1$$

$$G^{(2)}(x,p,z) = -\frac{i}{2} G^{(0)}(x,p) \star \partial_{p_\mu} Q(x,p) \star \partial_{p_\nu} G^{(0)}(x,p) F_{\mu\nu}(z)$$

Abelian axial current

Non – Abelian axial current

$$\langle J_\mu(x) \rangle = -\frac{1}{2} \mathrm{tr}_D \int \frac{d^4p}{(2\pi)^4} G_W \partial_{p_\mu} [Q, \gamma^5]$$

Response to (chromo) Magnetic field and μ in 3+1 D

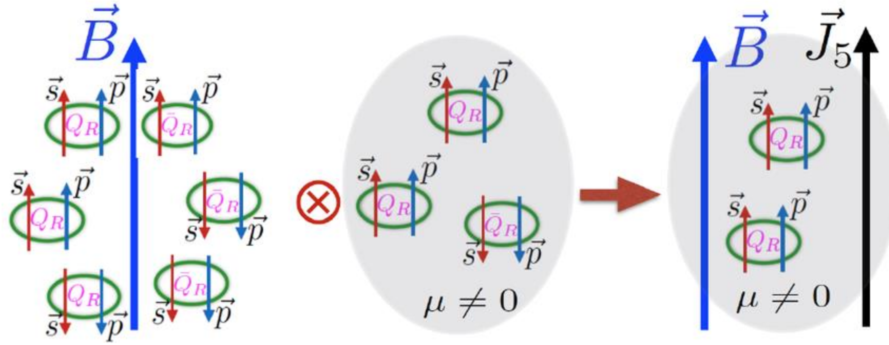
$$\frac{d}{d\mu} \bar{J}_i^{(5)} = \frac{1}{4\pi^2} \epsilon_{ijk} N_3 F_{jk}$$

$$N_3 = -\frac{1}{48\pi^2 V} \int d^3x \int_{\Sigma_0} \mathrm{tr}_D \left(G^{(0)} \star dQ \star \wedge dG^{(0)} \wedge dQ \right)$$

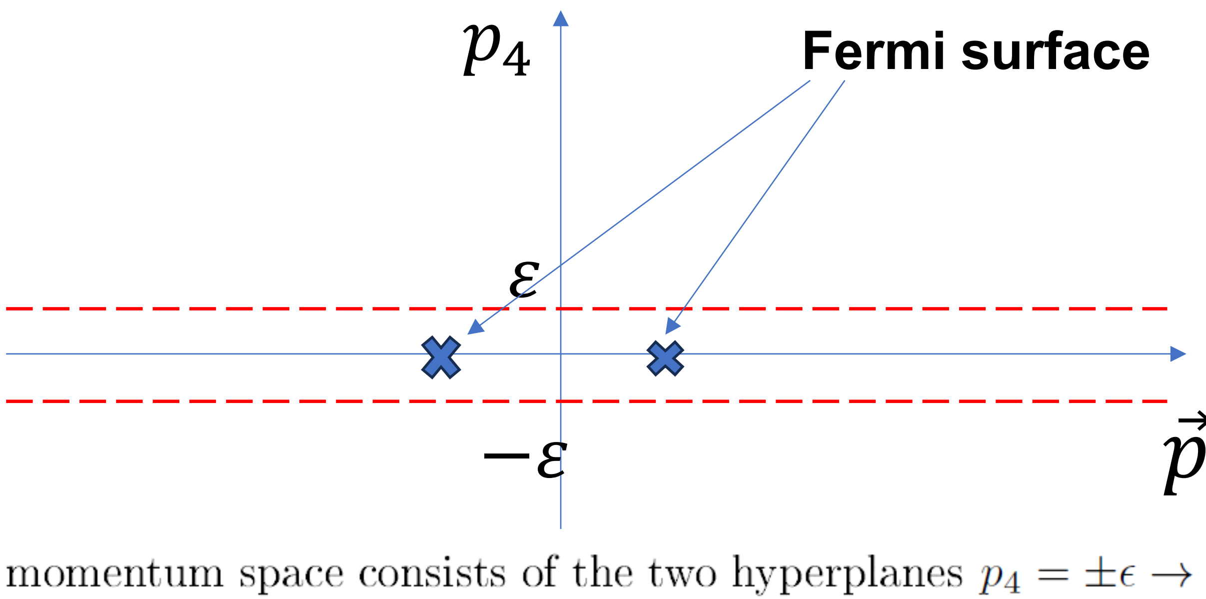
Σ_0 in $4D$ momentum space consists of the two hyperplanes $p_4 = \pm\epsilon \to 0$.

CHIRAL SEPARATION EFFECT

Axial current along magnetic field in the presence of chemical potential



Momentum space



Σ_0 in $4D$ momentum space consists of the two hyperplanes $p_4 = \pm\epsilon \to 0$.

Chiral anomaly vs. Atiyah – Singer theorem

$$Z = \int D\bar{\psi} D\psi \, e^{\int d^4x \, \bar{\psi}(x) Q \psi(x)}$$

$$Q = \begin{pmatrix} 0 & O^\dagger \\ O & 0 \end{pmatrix}$$

$$O = \sum_{|\alpha| \leq m} f_\alpha(x)(-i\partial)^\alpha$$

Principal symbol of operator O

$$o(x,p) := \sum_{|\alpha|=m} f_\alpha(x) p^\alpha$$

$$n_+ - n_- = \dim\ker O - \dim\ker O^\dagger = \operatorname{index} O$$

n_+ (resp. n_-) is defined as the number of zero modes of Q with positive (resp. negative) chirality

anomaly

$$\mathscr{A} := \int \langle \mathrm{tr} \mathcal{D}_\mu J_\mu \rangle = 2i(n_+ - n_-)$$

Atiyah – Singer theorem

$$\operatorname{index} O = \int d^4x d^4p \, \mathrm{ch}(\xi)(x,p) = \text{topological index } O$$

$$\mathscr{A} = 2i \int d^4x d^4p \, \mathrm{ch}(\xi)(x,p) = 2i \times \text{topological index } O$$

associated "virtual bundle" ξ

Chiral anomaly vs. Atiyah – Singer theorem

$$Z = \int D\bar{\psi} D\psi \, e^{\int d^4x \, \bar{\psi}(x) Q \psi(x)}$$

$$Q = \begin{pmatrix} 0 & O^\dagger \\ O & 0 \end{pmatrix}$$

$$O = \sum_{|\alpha| \le m} f_\alpha(x)(-i\partial)^\alpha$$

$$\mathscr{A} := \int \langle \mathrm{tr} \mathcal{D}_\mu J_\mu \rangle = 2i(n_+ - n_-)$$

n_+ (resp. n_-) is defined as the number of zero modes of Q with positive (resp. negative) chirality

For the fermions with conventional Dirac operator

$$\mathscr{A} = -\frac{i}{4\pi^2} \int \mathrm{tr} F \wedge F$$

In general case (obtained in our work for the first time)

$$\mathscr{A} = -N_3 \times \frac{i}{4\pi^2} \int \mathrm{tr} F \wedge F$$

$$N_3 := \frac{1}{48\pi^2 |V|} \int d^3\vec{x} \int_\Sigma \mathrm{tr}_D \left(\gamma^5 G^{(0)} \star dQ_W \star G^{(0)} \star \wedge dQ_W \star G^{(0)} \star \wedge dQ_W \right)$$

Provided that the topology in coordinate space Is due to the gauge field A only

Σ defined as the union of the two hyperplanes $p_4 = 0^\pm$

$$G^{(0)} \star Q_W = 1$$

Derivation

$$Z = \int D\bar{\psi} D\psi \, e^S$$

$$\text{with} \quad S = \int d^4x \, \bar{\psi}(x) Q(x, -iD) \psi(x)$$

$$Q(x, -iD) = \sum_{|\alpha| \leq m} c_\alpha(x)(-iD)^\alpha$$

$$S = -\mathrm{tr}_D \mathrm{tr}_G \mathrm{tr}_H \left(\hat{Q} \hat{\rho} \right)$$

$$\langle x | \hat{\rho} | y \rangle := \psi(x) \bar{\psi}(y)$$

Regularization: point splitting $\hat{\rho}^\epsilon := e^{i\hat{\pi}\epsilon} \hat{\rho} e^{i\hat{\pi}\epsilon}$

$$\langle x | \hat{\rho}^\epsilon | x \rangle = U(x, x+\epsilon) \psi(x+\epsilon) \psi(x-\epsilon) U(x-\epsilon, x)$$

$$\boxed{S^\epsilon = -\mathrm{tr}_D \mathrm{tr}_G \mathrm{tr}_H \left(\hat{Q} \hat{\rho}^\epsilon \right)}$$

Derivation

$$Z = \int D\bar{\psi} D\psi \, e^{S}$$

$$\boxed{S^{\epsilon} = -\mathrm{tr}_D \mathrm{tr}_G \mathrm{tr}_H \left(\hat{Q}\hat{\rho}^{\epsilon}\right)}$$

$$\langle x| \, \hat{\rho}^{\epsilon} \, |x\rangle = U(x, x+\epsilon)\psi(x+\epsilon)\bar{\psi}(x-\epsilon)U(x-\epsilon, x)$$

Noether current corresponding to chiral transformation

$$\psi(x) \to e^{i\alpha(x)\gamma^5}\psi(x)$$
$$\bar{\psi}(x) \to \bar{\psi}(x) e^{i\alpha(x)\gamma^5}$$

$$\alpha \in \mathfrak{g}$$

Variation of action

$$\delta S^{\epsilon} = -i\mathrm{tr}_D \mathrm{tr}_G \mathrm{tr}_H \left(\alpha(\hat{x})\gamma^5\{\hat{Q}, \hat{\rho}^{\epsilon}\}\right)$$

$$\delta S^{\epsilon} = \mathrm{tr}_G \int d^4x \, \alpha(x)\Gamma^{\epsilon}(x)$$

$$\Gamma^{\epsilon}(x) = -i\mathrm{tr}_D \gamma^5 \int \frac{d^4p}{(2\pi)^4} (Q_W \star \rho^{\epsilon}_W + \rho^{\epsilon}_W \star Q_W)$$

$$\rho^{\epsilon}_W = e^{ip\epsilon} \star \rho_W \star e^{ip\epsilon}$$

$$\Gamma^{\epsilon}(x) = \mathcal{D}_{\mu} J^{\epsilon}_{\mu}(x)$$

axial current: higher orders in derivatives

$$J^{\epsilon}_{\mu}(x) := -\frac{1}{2}\mathrm{tr}_D \gamma^5 \int \frac{d^4p}{(2\pi)^4} \left(\partial_{p_\mu} Q_W(x,p)\rho^{\epsilon}_W(x,p) - \rho^{\epsilon}_W(x,p)\partial_{p_\mu} Q_W(x,p)\right) + \ldots$$

Derivation

$$Z = \int D\bar{\psi} D\psi \, e^{S}$$

$$\boxed{S^{\epsilon} = -\mathrm{tr}_D \mathrm{tr}_G \mathrm{tr}_H \left(\hat{Q}\hat{\rho}^{\epsilon}\right)}$$

$$\psi(x) \to e^{i\alpha(x)\gamma^5}\psi(x)$$
$$\bar{\psi}(x) \to \bar{\psi}(x) e^{i\alpha(x)\gamma^5}$$

$$\delta S^{\epsilon} = \mathrm{tr}_G \int d^4x \, \alpha(x) \Gamma^{\epsilon}(x)$$

$$\Gamma^{\epsilon}(x) = \mathcal{D}_{\mu} J^{\epsilon}_{\mu}(x)$$

axial current: higher orders in covariant derivatives

$$J^{\epsilon}_{\mu}(x) := -\frac{1}{2}\mathrm{tr}_D \gamma^5 \int \frac{d^4p}{(2\pi)^4} \left(\partial_{p_\mu} Q_W(x,p) \rho^{\epsilon}_W(x,p) - \rho^{\epsilon}_W(x,p) \partial_{p_\mu} Q_W(x,p)\right) + \dots$$

Chiral anomaly:

$$\mathrm{tr}_G \langle \mathcal{D}_{\mu} J^{\epsilon}_{\mu} \rangle = i \mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4p \left(Q_W \star e^{ip\epsilon} \star G_W \star e^{ip\epsilon} + e^{ip\epsilon} \star G_W \star e^{ip\epsilon} \star Q_W \right)$$

With extra integration over x we have a divergent expression ➔ infrared regularization (integration over a finite region of space)

Expansion in powers of F: sum of $\sim e^{2i\epsilon p} \epsilon^n F^m$ with $m \geq n$

The terms with n> 1 are irrelevant in the limit $\epsilon \to 0$

$$\int d^4x \, \mathrm{tr}_G \langle \mathcal{D}_{\mu} J^{\epsilon}_{\mu} \rangle = -2i \, \mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4x d^4p \, e^{2ip\epsilon} \epsilon_{\mu} \left(\partial_{x_\mu} Q_W - F_{\mu\nu} \partial_{p_\nu} Q_W + \frac{1}{24} \mathcal{D}_{\alpha} \mathcal{D}_{\beta} F_{\mu\nu} \partial_{p_\alpha} \partial_{p_\beta} \partial_{p_\nu} Q_W \right) G_W$$

Chiral anomaly: $$Z = \int D\bar{\psi} D\psi \, e^S$$

$$\mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = i \mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4p \left(Q_W \star e^{ip\epsilon} \star G_W \star e^{ip\epsilon} + e^{ip\epsilon} \star G_W \star e^{ip\epsilon} \star Q_W \right)$$

Up to the terms, which do not disappear in the limit $\epsilon \to 0$

$$\int d^4x \, \mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = -2i \, \mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4x d^4p \, e^{2ip\epsilon} \epsilon_\mu \left(\partial_{x_\mu} Q_W - F_{\mu\nu} \partial_{p_\nu} Q_W + \frac{1}{24} \mathcal{D}_\alpha \mathcal{D}_\beta F_{\mu\nu} \partial_{p_\alpha} \partial_{p_\beta} \partial_{p_\nu} Q_W \right) G_W$$

We use the theorem (averaging over directions)

$$\lim_{|\epsilon| \to 0} \left\langle \int d^4p \, e^{ip\epsilon} \epsilon_\mu f(p) \right\rangle = i \int d^4p \, \partial_\mu f(p)$$

$$\lim_{|\epsilon| \to 0} \int d^4x \, \mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = + \mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4x d^4p \, \partial_{p_\mu} \left(\left(\partial_{x_\mu} Q_W - F_{\mu\nu} \partial_{p_\nu} Q_W + \frac{1}{24} \mathcal{D}_\alpha \mathcal{D}_\beta F_{\mu\nu} \partial_{p_\alpha} \partial_{p_\beta} \partial_{p_\nu} Q_W \right) G_W \right)$$

Topology in coordinate space is due to the gauge field only

$$\updownarrow$$

$\boxed{Q_W(x,p) \text{ is homotopic to a function } \tilde{Q}(p)}$ $\qquad \tilde{G}_W \star \tilde{Q} = 1$

$$\mathscr{A} = -\mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4x d^4p \, \partial_{p_\mu} \left((F_{\mu\nu} \partial_{p_\nu} \tilde{Q} - \frac{1}{24} \mathcal{D}_\alpha \mathcal{D}_\beta F_{\mu\nu} \partial_{p_\alpha} \partial_{p_\beta} \partial_{p_\nu} \tilde{Q}) \tilde{G}_W \right)$$

$$\tilde{G}_W = \tilde{G}^{(0)} + \frac{i}{2} \tilde{G}^{(0)} \partial_{p_\alpha} \tilde{Q} \, \tilde{G}^{(0)} \partial_{p_\beta} \tilde{Q} \, \tilde{G}^{(0)} F_{\alpha\beta} + O(F^2)$$

Chiral anomaly:

$$\lim_{|\epsilon|\to 0}\int d^4x\,\mathrm{tr}_G\langle \mathcal{D}_\mu J^\epsilon_\mu\rangle = +\mathrm{tr}_D\mathrm{tr}_G\gamma_5\int (2\pi)^{-4}d^4x d^4p\,\partial_{p_\mu}\left(\left(\partial_{x_\mu}Q_W - F_{\mu\nu}\partial_{p_\nu}Q_W + \frac{1}{24}\mathcal{D}_\alpha\mathcal{D}_\beta F_{\mu\nu}\partial_{p_\alpha}\partial_{p_\beta}\partial_{p_\nu}Q_W\right)G_W\right)$$

Topology in coordinate space is due to the gauge field only

$$\boxed{Q_W(x,p)\ \text{is homotopic to a function}\ \tilde{Q}(p)} \qquad \tilde{G}_W \bigstar \tilde{Q} = 1$$

$$\mathscr{A} = -\mathrm{tr}_D\mathrm{tr}_G\gamma_5\int (2\pi)^{-4}d^4x d^4p\,\partial_{p_\mu}\left((F_{\mu\nu}\partial_{p_\nu}\tilde{Q} - \frac{1}{24}\mathcal{D}_\alpha\mathcal{D}_\beta F_{\mu\nu}\partial_{p_\alpha}\partial_{p_\beta}\partial_{p_\nu}\tilde{Q})\tilde{G}_W\right)$$

$$\tilde{G}_W = \tilde{G}^{(0)} + \frac{i}{2}\tilde{G}^{(0)}\partial_{p_\alpha}\tilde{Q}\,\tilde{G}^{(0)}\partial_{p_\beta}\tilde{Q}\,\tilde{G}^{(0)}F_{\alpha\beta} + O(F^2)$$

$$\mathscr{A} = -2iN_3\int \frac{1}{16\pi^2}d^4x\,\mathrm{tr}(FF^\star)$$

$$N_3 = \frac{1}{8\pi^2}\int dS$$

$$S_{\alpha\beta\nu}(x) := \frac{1}{2}\mathrm{tr}_D\left(\gamma^5\tilde{G}^{(0)}\partial_{p_\alpha}\tilde{Q}\,\tilde{G}^{(0)}\partial_{p_\beta}\tilde{Q}\,\tilde{G}^{(0)}\partial_{p_\nu}\tilde{Q}\right) - (\alpha\leftrightarrow\beta)$$

Chiral anomaly:

$$Z = \int D\bar{\psi} D\psi \, e^S$$

$$\mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = i \,\mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4 p \left(Q_W \bigstar e^{ip\epsilon} \bigstar G_W \bigstar e^{ip\epsilon} + e^{ip\epsilon} \bigstar G_W \bigstar e^{ip\epsilon} \bigstar Q_W \right)$$

Up to the terms, which do not disappear in the limit $\epsilon \to 0$

$$\int d^4x \,\mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = -2i \,\mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4x d^4p \, e^{2ip\epsilon} \epsilon_\mu \left(\partial_{x_\mu} Q_W - F_{\mu\nu} \partial_{p_\nu} Q_W + \frac{1}{24} \mathcal{D}_\alpha \mathcal{D}_\beta F_{\mu\nu} \partial_{p_\alpha} \partial_{p_\beta} \partial_{p_\nu} Q_W \right) G_W$$

We use the theorem (averaging over directions)

$$\lim_{|\epsilon| \to 0} \left\langle \int d^4p \, e^{ip\epsilon} \epsilon_\mu f(p) \right\rangle = i \int d^4p \, \partial_\mu f(p)$$

$$\lim_{|\epsilon| \to 0} \int d^4x \,\mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = + \mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4x d^4p \, \partial_{p_\mu} \left(\left(\partial_{x_\mu} Q_W - F_{\mu\nu} \partial_{p_\nu} Q_W + \frac{1}{24} \mathcal{D}_\alpha \mathcal{D}_\beta F_{\mu\nu} \partial_{p_\alpha} \partial_{p_\beta} \partial_{p_\nu} Q_W \right) G_W \right)$$

Topology in coordinate space is due to the gauge field only

$Q_W(x,p)$ is homotopic to a function $\widetilde{Q}(p)$

$$\mathscr{A} = -2i N_3 \int \frac{1}{16\pi^2} d^4x \,\mathrm{tr}(F F^\star)$$

$$N_3 = \frac{1}{8\pi^2} \int_\Sigma S = \frac{1}{48\pi^2} \int_\Sigma \mathrm{tr}_D \left(\gamma^5 \widetilde{G}^{(0)} d\widetilde{Q} \, \widetilde{G}^{(0)} \wedge d\widetilde{Q} \widetilde{G}^{(0)} \wedge d\widetilde{Q} \right)$$

Chiral anomaly:

$$Z = \int D\bar{\psi} D\psi \, e^{S}$$

$$\mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = i \mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4 p \left(Q_W \bigstar e^{ip\epsilon} \bigstar G_W \bigstar e^{ip\epsilon} + e^{ip\epsilon} \bigstar G_W \bigstar e^{ip\epsilon} \bigstar Q_W \right)$$

Up to the terms, which do not disappear in the limit $\epsilon \to 0$

$$\int d^4x \, \mathrm{tr}_G \langle \mathcal{D}_\mu J^\epsilon_\mu \rangle = -2i \, \mathrm{tr}_D \mathrm{tr}_G \gamma_5 \int (2\pi)^{-4} d^4x d^4p \, e^{2ip\epsilon} \epsilon_\mu \left(\partial_{x_\mu} Q_W - F_{\mu\nu} \partial_{p_\nu} Q_W + \frac{1}{24} \mathcal{D}_\alpha \mathcal{D}_\beta F_{\mu\nu} \partial_{p_\alpha} \partial_{p_\beta} \partial_{p_\nu} Q_W \right) G_W$$

We use the theorem (averaging over directions)

$$\lim_{|\epsilon| \to 0} \left\langle \int d^4p \, e^{ip\epsilon} \epsilon_\mu f(p) \right\rangle = i \int d^4p \, \partial_\mu f(p)$$

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Topology in coordinate space is due to the gauge field only

$\Updownarrow$

$Q_W(x,p)$ is homotopic to a function $\widetilde{Q}(p)$

$$\mathscr{A} = -2iN_3 \int \frac{1}{16\pi^2} d^4x \, \mathrm{tr}(F F^\star)$$

$$N_3 = \frac{1}{48\pi^2 |V|} \int d^3\vec{x} \int_\Sigma \mathrm{tr}_D \left(\gamma^5 G^{(0)} \star dQ_W \star G^{(0)} \star \wedge dQ_W \star G^{(0)} \star \wedge dQ_W \right)$$

Chiral anomaly: $$Z = \int D\bar{\psi} D\psi \, e^S$$

Topology in coordinate space is due to the gauge field only

$$\boxed{Q_W(x,p) \text{ is homotopic to a function } \widetilde{Q}(p)}$$

$$\mathscr{A} := \int \langle \mathrm{tr} \mathcal{D}_\mu J_\mu \rangle$$

$$\boxed{\mathscr{A} = -2iN_3 \int \frac{1}{16\pi^2} d^4x \, \mathrm{tr}(FF^\star)}$$

$$\boxed{N_3 = \frac{1}{48\pi^2 |V|} \int d^3\vec{x} \int_\Sigma \mathrm{tr}_D \left(\gamma^5 G^{(0)} \star dQ_W \star G^{(0)} \star \wedge dQ_W \star G^{(0)} \star \wedge dQ_W \right)}$$

In Minkowski space – time:

$$\mathscr{A} = N_3 \times \frac{1}{2\pi^2} \int d^4x \, \mathrm{tr}(\mathbf{E}.\mathbf{B})$$

Chiral anomaly:

$$Z = \int D\bar{\psi} D\psi \, e^S$$

Topology in coordinate space is due to the gauge field only

$$\Updownarrow$$

$$\boxed{Q_W(x,p) \text{ is homotopic to a function } \widetilde{Q}(p)}$$

$$\mathscr{A} := \int \langle \mathrm{tr} \mathcal{D}_\mu J_\mu \rangle$$

$$\mathscr{A} = 2i \int d^4x d^4p \, \mathrm{ch}(\xi)(x,p) = 2i \times \text{topological index } O$$

$$\boxed{\mathscr{A} = -2iN_3 \int \frac{1}{16\pi^2} d^4x \, \mathrm{tr}(FF^\star)}$$

$$\boxed{N_3 = \frac{1}{48\pi^2|V|} \int d^3\vec{x} \int_\Sigma \mathrm{tr}_D \left(\gamma^5 G^{(0)} \star dQ_W \star G^{(0)} \star \wedge dQ_W \star G^{(0)} \star \wedge dQ_W \right)}$$

In Minkowski space – time:

$$\mathscr{A} = N_3 \times \frac{1}{2\pi^2} \int d^4x \, \mathrm{tr}(\mathbf{E}.\mathbf{B})$$

Example

$$Z = \int D\bar{\psi} D\psi \, e^{S}$$

$$\hat{Q} = \begin{pmatrix} 0 & \hat{O}^\dagger \\ \hat{O} & 0 \end{pmatrix}$$

$$\hat{O} = \hat{\pi}_4 + i \begin{pmatrix} \hat{\pi}_3 & \kappa(\hat{\pi}_1 - i\hat{\pi}_2)^n \\ \kappa(\hat{\pi}_1 + i\hat{\pi}_2)^n & -\hat{\pi}_3 \end{pmatrix}$$

$$\mathscr{A} := \int \langle \mathrm{tr} \mathcal{D}_\mu J_\mu \rangle$$

$$\mathscr{A} = 2i \int d^4x d^4p \, \mathrm{ch}(\xi)(x,p) = 2i \times \text{topological index } O$$

Topology in coordinate space is due to the gauge field only

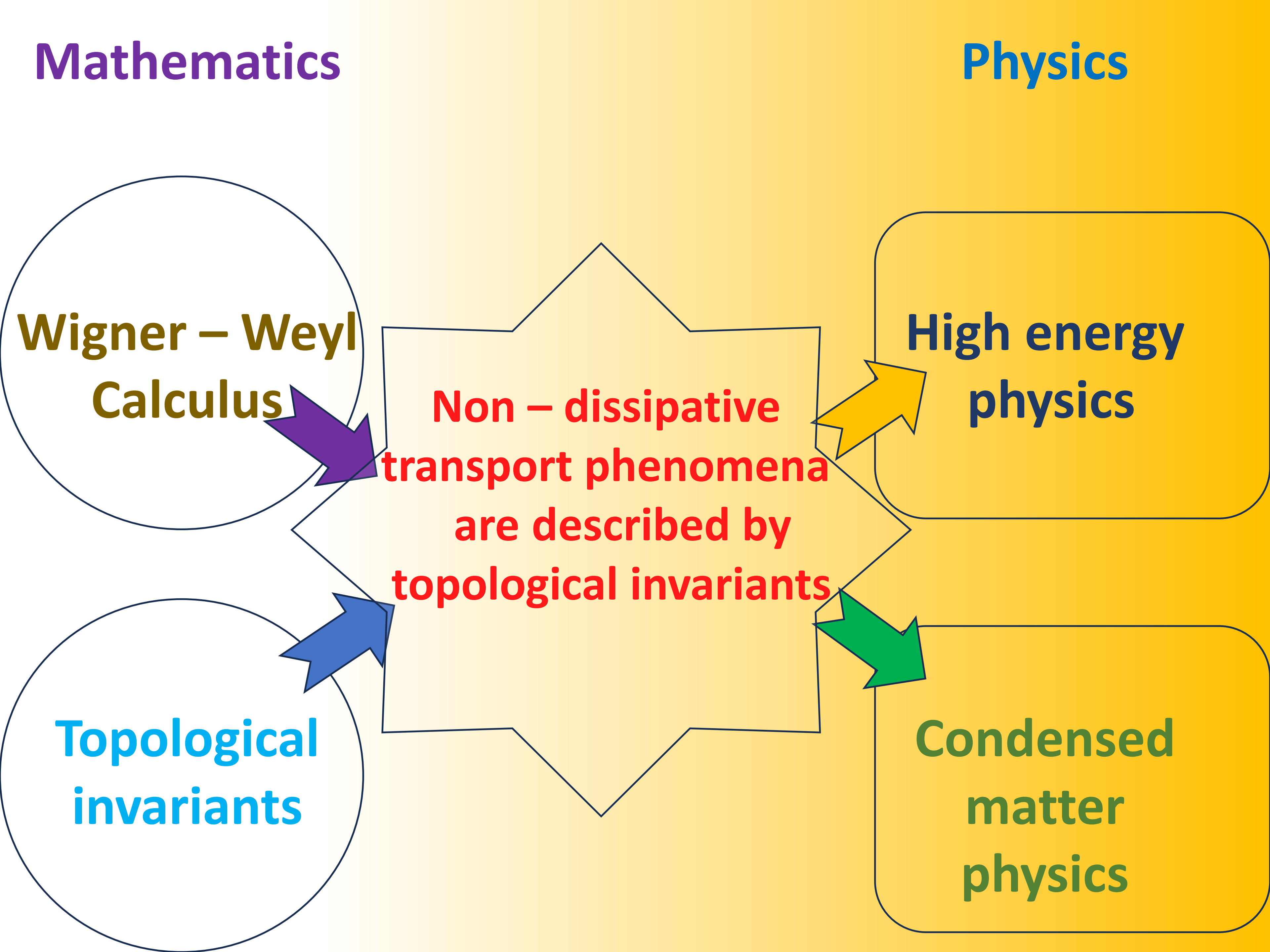
$$\mathscr{A} = -2iN_3 \int \frac{1}{16\pi^2} d^4x \, \mathrm{tr}(FF^\star)$$

$$\boxed{N_3 = n}$$

$$N_3 = \frac{1}{48\pi^2 |V|} \int d^3\vec{x} \int_\Sigma \mathrm{tr}_D \left(\gamma^5 G^{(0)} \star dQ_W \star G^{(0)} \star \wedge dQ_W \star G^{(0)} \star \wedge dQ_W \right)$$

In Minkowski space – time:

$$\mathscr{A} = N_3 \times \frac{1}{2\pi^2} \int d^4x \, \mathrm{tr}(\mathbf{E}.\mathbf{B})$$


Mathematics
Physics
Wigner – Weyl Calculus
Topological invariants
Non – dissipative transport phenomena are described by topological invariants
High energy physics
Condensed matter physics

Conclusions

- Covariant Wigner – Weyl calculus allows to represent in compact form the conductivities of non – dissipative transport phenomena in non – uniform systems.
- We consider in this respect the non – Abelian versions of quantum Hall effect and chiral separation effect. Their conductivities are the same as for their Abelian versions.
- Chiral anomaly is equal to the product of the topological invariant responsible for the CSE and the number of instantons. This may have experimental consequences if Dirac operator is not linear in momentum in certain circumstances.