Lagrangian Formulation of General Relativity

Student: Raul de Assis Santos[1, 2]
Supervisor: Bertha M. Cuadros-Melgar[2]
Luis H. V. Melo[1]

[1]Instituto Superior Tecnico (IST)
[2]Escola de Engenharia de Lorena (EEL-USP)

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Presentation Structure

Conventions

The unities will be adjusted so that $c = 1$. The signature of the metric $(-, +, +, +)$.

The Lagrangian density

$$L = \int d^n x \, \mathcal{L}(\phi, \partial_\mu \phi) \, ,$$

$$\mathcal{L} \equiv \sqrt{-g} \tilde{\mathcal{L}} \, ,$$

where ϕ is the scalar field under study and $\partial_\mu \phi \equiv \partial \phi / \partial x^\mu$.

The notation used to express covariant derivatives:

$$\nabla_\mu V^\nu \equiv \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\alpha} V^\alpha \, .$$

The action S, expressed in terms of the field

$$S = \int dt \, L(\phi, \partial_\mu \phi)).$$

$$\frac{\delta S}{\delta \phi} = 0$$

Einstein Equations

In order to obtain Einstein Equations, we begin with the following action:

$$S_{EH} = \int d^4x \sqrt{-g} R \,.$$

First, for the variation of $\sqrt{-g}$

$$\delta\left(\sqrt{-g}\right) = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu} \,.$$

From the definition of the Ricci scalar:

$$\delta R = g^{\mu\nu}\delta R_{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu} \,.$$

The variation of the Ricci tensor:

$$\delta R^{\lambda}{}_{\mu\lambda\nu} = \nabla_\lambda\left(\delta\Gamma^{\lambda}{}_{\nu\mu}\right) - \nabla_\nu\left(\delta\Gamma^{\lambda}{}_{\lambda\mu}\right) \,.$$

Applying these results:

$$\delta S_{EH} = \int d^4x \sqrt{-g}\left(-\frac{1}{2} R g_{\mu\nu}\right)\delta g^{\mu\nu} + \int d^4x \sqrt{-g}\left(R_{\mu\nu}\right)\delta g^{\mu\nu} + \cancel{\int d^4x \sqrt{-g}\, g^{\mu\nu}\delta R_{\mu\nu}} \,,$$

the last term was canceled since it can be reduced to a surface term.

Einstein Equations

The Einstein equations in vacuum:

$$\therefore \frac{1}{\sqrt{-g}}\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0.$$

To obtain the complete Einstein Equations it is necessary to take an action from matter S_M into account.

Using the total action S

$$S = \frac{1}{16\pi G}S_{EH} + S_M,$$

by defining,

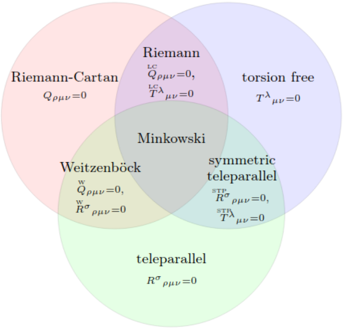
$$\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = -\frac{1}{2}T_{\mu\nu},$$

We finally obtain

$$\frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) - \frac{1}{2}T_{\mu\nu} = 0\,,$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \quad \blacksquare$$

Modified Gravity



Venn diagram from [1]

New Massive Gravity

From de Rham [2]

- GR may be considered a theory for a spin-2 massless particle.
- It should be possible to construct an invariant theory of massive gravity in 3 dimensions that has the same degrees of freedom that a massless gravity in four dimensions.

NMG is defined by supplementing Einstein-Hilbert action with particular square-curvature terms.

Results from Ayón-Beato

Ayón-Beato et al.[3] developed an important contribution in this direction.

We begin with the action of NMG given by,

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left[R - 2\lambda - \frac{1}{m^2}\left(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\right)\right] .$$

With the variation

$$\delta S = \frac{1}{16\pi G}\delta\int d^3x\sqrt{-g}(R-2\lambda) - \frac{1}{16\pi G}\frac{1}{m^2}\delta\int d^3x\sqrt{-g}\left(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2\right) .$$

Results from Ayón-Beato

We used the term

$$K_{\mu\nu} = 2\Box R_{\mu\nu} - \frac{1}{2}\nabla_\mu \nabla_\nu R - \frac{1}{2}\Box R g_{\mu\nu} + 4 R_{\mu\alpha\nu\beta} R^{\alpha\beta} - \frac{3}{2} R R_{\mu\nu} - R_{\alpha\beta} R^{\alpha\beta} g_{\mu\nu} + \frac{3}{8} R^2 g_{\mu\nu}\,,$$

here, $G = 1/8$ was used for simplicity.
Finally, the corresponding variation of the action reads,

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \lambda g_{\mu\nu} - \frac{1}{2m^2} K_{\mu\nu} = 0\,,$$

Results from Ayón-Beato

Using the *Ansatz*,

$$ds^2 = -\frac{r^{2z}}{l^{2z}} F(r)\, dt^2 + \frac{l^2}{r^2} H(r)\, dr^2 + \frac{r^2}{l^2} d\vec{x}^2 \,,$$

for z=3.

And by adjusting the boundary conditions

$$F(r_+) = H^{-1}(r_+) = 0 \quad \text{and,} \quad \lim_{r\to\infty} F(r) = \lim_{r\to\infty} H(r)^{-1} = 1 \,,$$

The functions $F(r)$ and $H(r)$ were obtained

$$F(r) = 1 - \frac{Ml^2}{r^2} \qquad H(r) = F(r)^{-1} = -\frac{r^2}{Ml^2 - r^2} \,.$$

Invariants and Horizon

It was determined that the horizon is located at $r_+ = l\sqrt{M}$.

It was also possible to calculate the invariants R and $R_{\mu\nu}R^{\mu\nu}$ with the software *Maple 18*[4]

$$R = -\frac{26}{l^2} + \frac{8M}{r^2}, \qquad R_{\mu\nu}R^{\mu\nu} = \frac{24M^2}{r^4} + \frac{260}{l^4} - \frac{152M}{r^2 l^2}.$$

Same horizon found in Bañados-Teitelboim-Zanelli (BTZ) black hole (a GR solution) [5].

Conclusion

- Lagrangian formulation offers a powerful and elegant framework for understanding gravitational interactions in terms of a variational principle.
- It was possible to obtain a geometry describing an $r = 0$ singularity covered by an event horizon located at the same position of the GR BTZ solution.
- Exploring Modified Gravity highlights the potential for extending our understanding of gravitational phenomena beyond classical GR.

Perspectives

- Expand the studies of the AGGH (thermodynamics, quasinormal modes...)
- Explore other Lifshitz Points
- AdS/CFT conjecture

References I

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References II

[4] Maplesoft. *Maple 18*. Maplesoft, a division of Waterloo Maple Inc., Waterloo, Ontario. 2014. URL: https://www.maplesoft.com.

[5] Máximo Bañados, Claudio Teitelboim e Jorge Zanelli. "Black hole in three-dimensional spacetime". Em: *Phys. Rev. Lett.* 69 (13 1992), pp. 1849–1851. DOI: 10.1103/PhysRevLett.69.1849. URL: https://link.aps.org/doi/10.1103/PhysRevLett.69.1849.