TOPOLOGICAL MODES IN 2D MAGNETIZED DIRAC MATERIALS

2nd Cycle Integrated Project in Engineering Physics

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Vasco, what's really your project?

How am I approaching the problem?

- The problem is approached both theoretically, and numerically;
- The basis for this is an hydrodynamical model for the electron liquid;

What have I done already?

- Utilized a simplified, linearized model;
- Obtained good analytical results for interface modes;
- Numerical simulations show good agreement with the theoretical predictions;

What do I plan to do in the future?

- Improve the already existing framework for simulations;
- Attempt to arrive at new analytical solutions, using more comprehensive models, reaching new physics;

How am I doing it?

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{v}) = 0$$
$$\frac{\mathcal{D}}{\mathcal{D}t}(nm\vec{v}) + \nabla \cdot \mathbf{\Pi} + en\nabla\phi + \Omega_B \vec{v} \times \hat{z} = 0$$

System of 3 equations for the fields $[n, u, v]$

$$m = \frac{\hbar\sqrt{\pi n}}{v_F}$$

$$\vec{v} = (u, v)$$

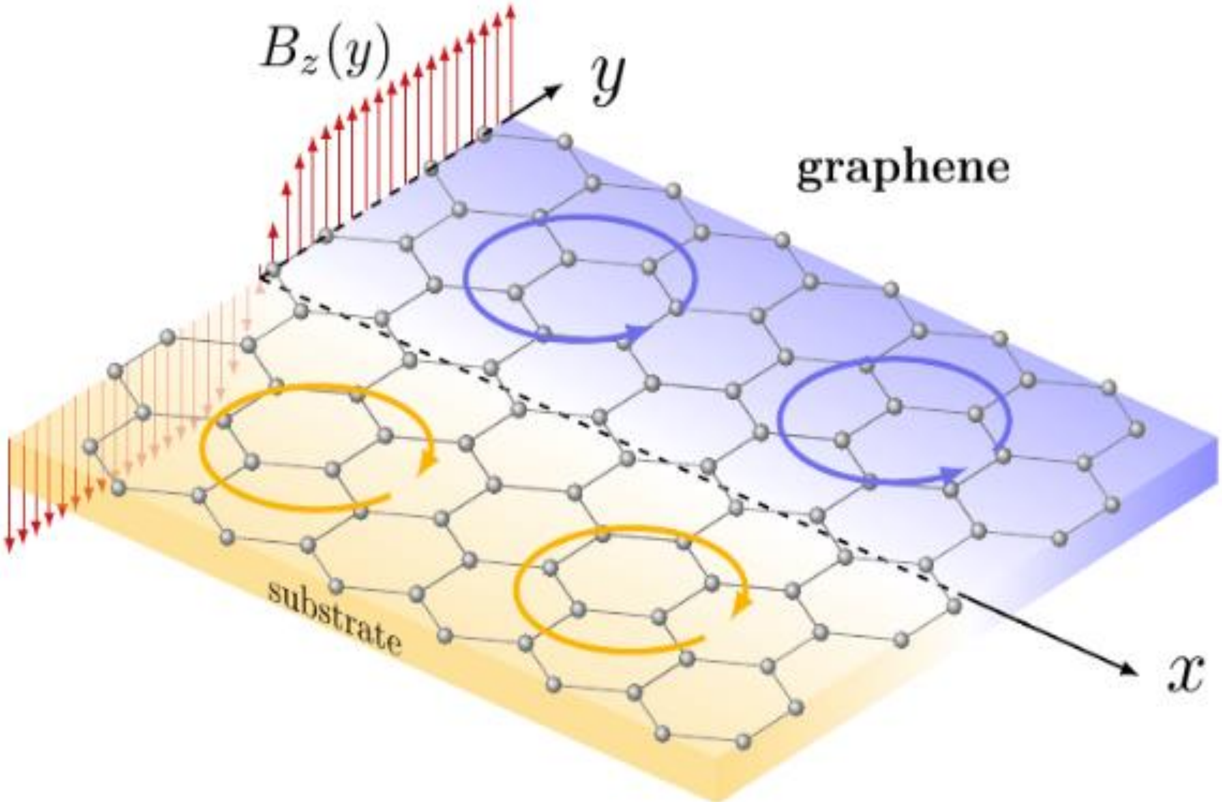
Cyclotron frequency: $\Omega_B(y) = \frac{qB(y)}{m}$

Electrostatic potential: $\nabla\phi = \frac{ed_0}{\varepsilon}\nabla n$

Stress tensor: $\nabla \cdot \Pi = \nabla P_F - \eta_o \nabla^2 (\vec{v} \times \hat{z})$

Full time derivative: $\frac{\mathcal{D}}{\mathcal{D}t} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$

How am i doing it?



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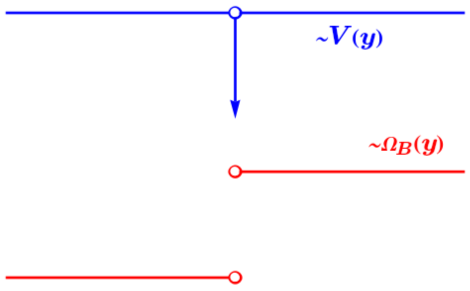
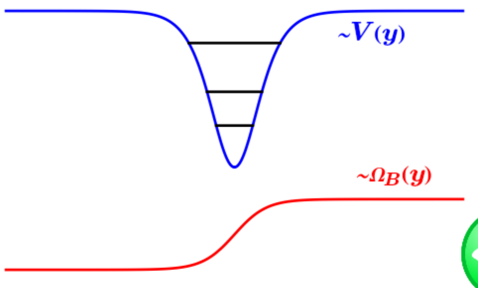
How am I doing it?



Sharp transition

$$\Omega_B(y) = \Omega_B \operatorname{sgn}(y)$$

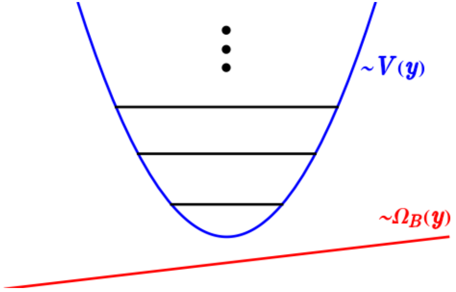
- Only allows one bound state;
- Discontinuous function;
- May lead to non-physical results;



Smooth transition

$$\Omega_B(y) = \Omega_B \tanh\left(\frac{y}{\epsilon}\right)$$

- Allows a finite number of bound states;
- Continuous function, assures continuity of solutions;
- Other two functions can be seen as limits of this one;



Linear transition

$$\Omega_B(y) = \frac{\Omega_B}{\epsilon} y$$

- Allows an infinite number of bound states;
- Does not contemplate the stabilization of the magnetic field;
- Could only be useful for a very localized approximation;

What have I done?

Linearizing the equations yields:

$$\frac{\partial n}{\partial t} + n_0 \frac{\partial u}{\partial x} + n_0 \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + \frac{S^2}{n_0} \frac{\partial n}{\partial x} - \Omega_B v - \eta_o \nabla^2 v = 0$$

$$\frac{\partial v}{\partial t} + \frac{S^2}{n_0} \frac{\partial n}{\partial y} + \Omega_B u + \eta_o \nabla^2 u = 0$$

Hall Viscosity: η_o

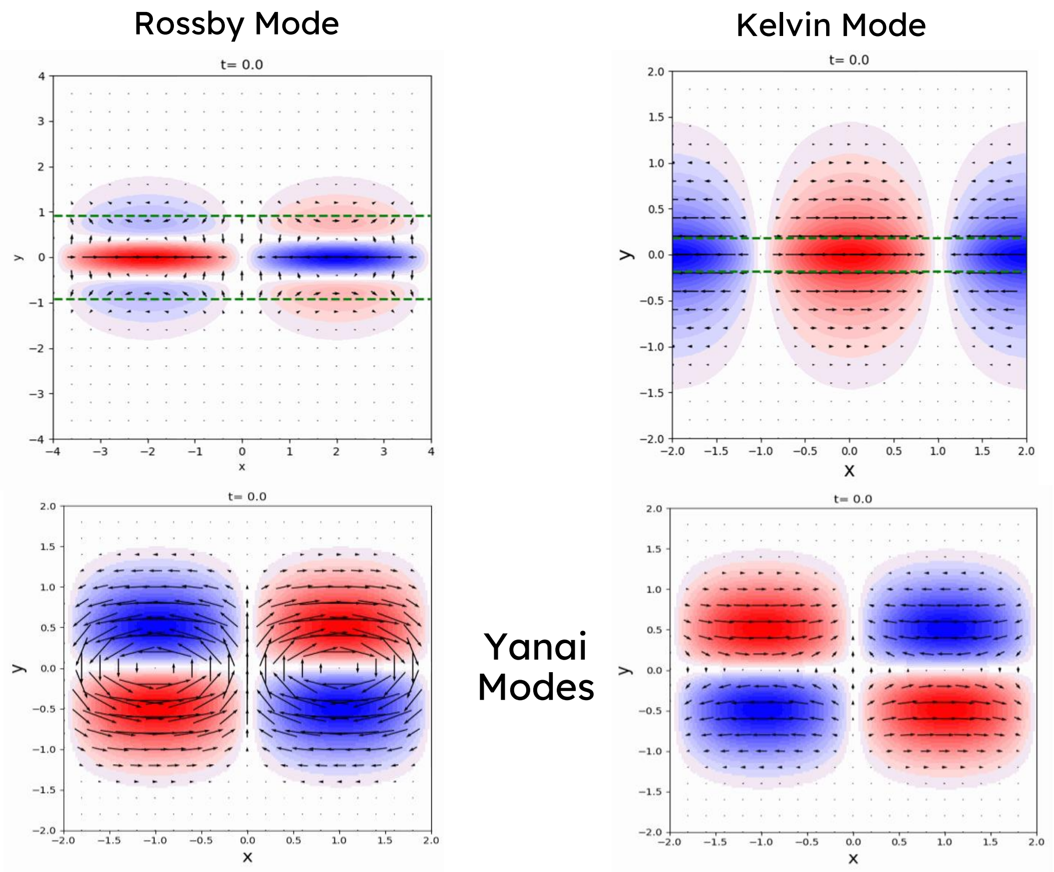
Cyclotron frequency: Ω_B

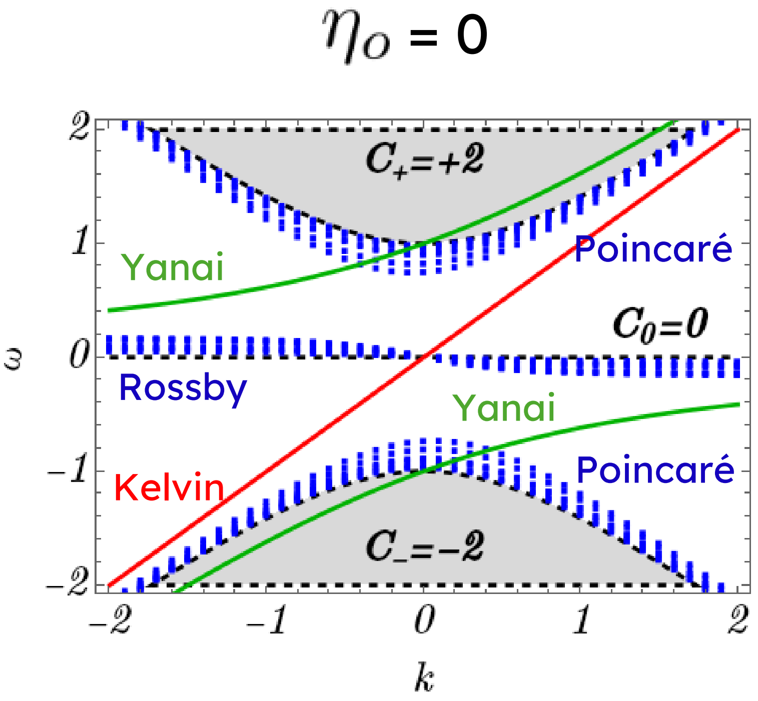
Equilibrium density: n_0

Plasmon sound velocity: $S = e\sqrt{\frac{d_0 n_0}{\varepsilon m_0}}$

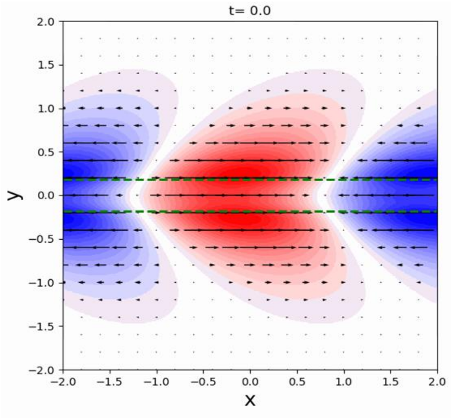
The only term that is retained from the stress tensor is the Hall viscosity term

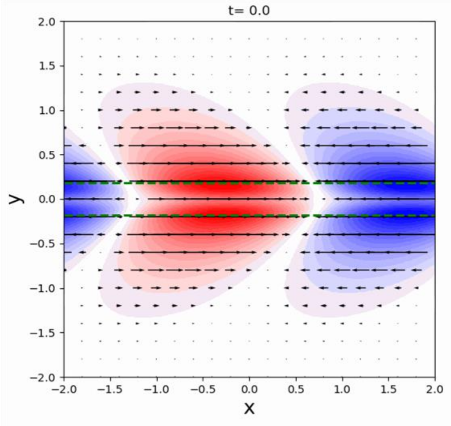
What have I got?



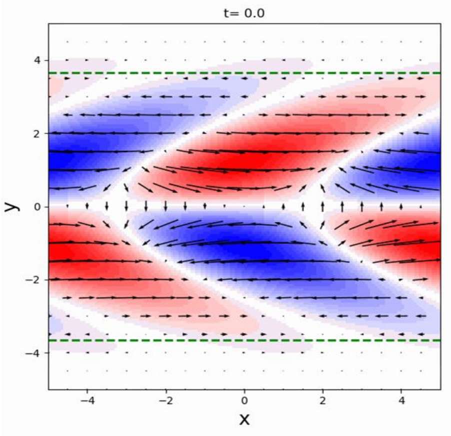


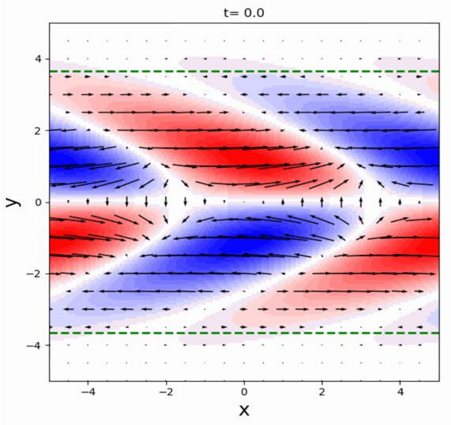
What have I got?





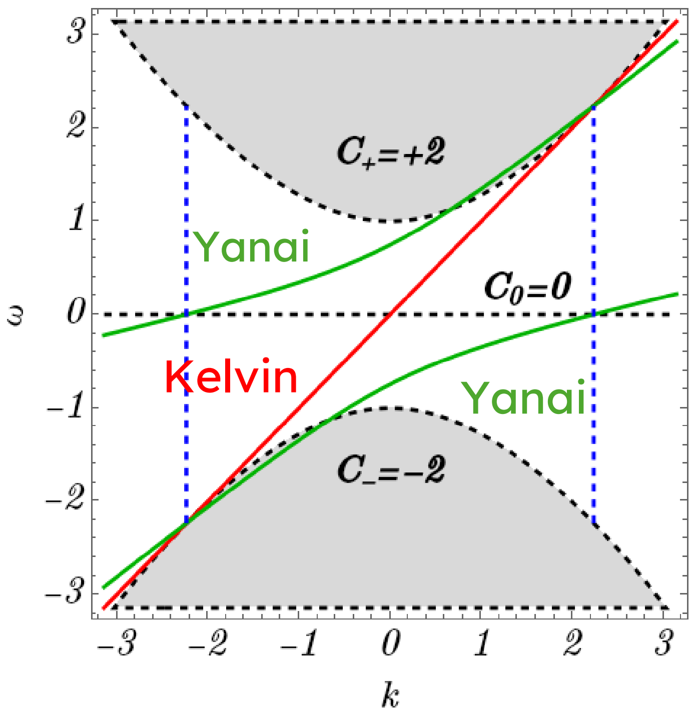
Kelvin Modes



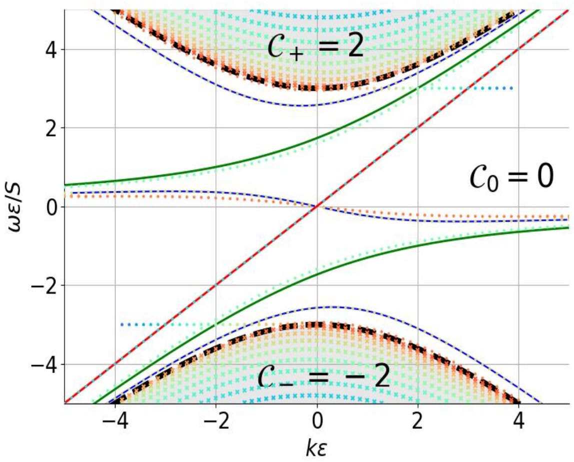


Yanai Modes

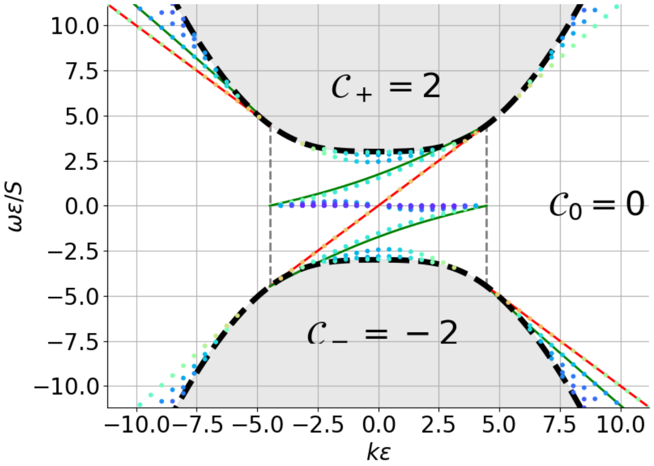
$\eta_o \neq 0$



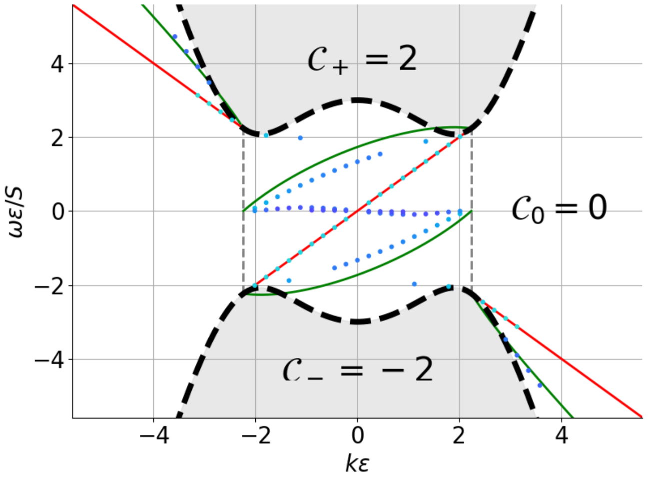
$\eta_\circ = 0$



$\eta_\circ = 0.15$



$\eta_\circ = 0.6$



If you were not convinced, yes, there is a very good agreement between the theory, and the numerical results.

What will I do?

- Where are the Rossby modes, when considering the Hall viscosity?
  - Ans: They could exist as shown by numerical results, but more analytical work needs to be performed.

- Isn't the linearized model a very simplified regime?
  - Ans: Yes, and in the future, nonlinearities must be considered to have a more complete understanding of the system.

- What can be improved in the numerical simulations?
  - Ans: So far, some extra considerations regarding time scaling need to be reviewed, which that can be crucial to understand how the modes evolve.

Questions?

References

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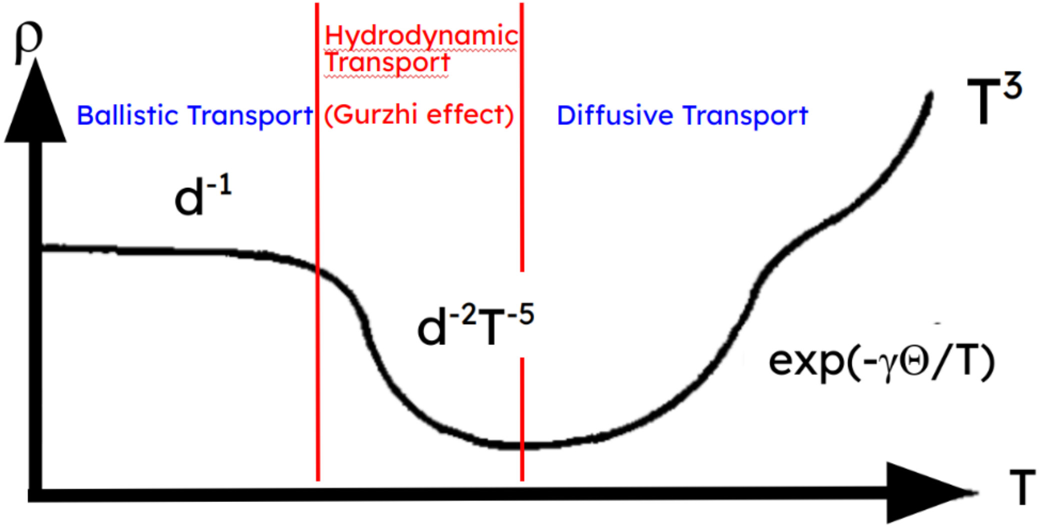
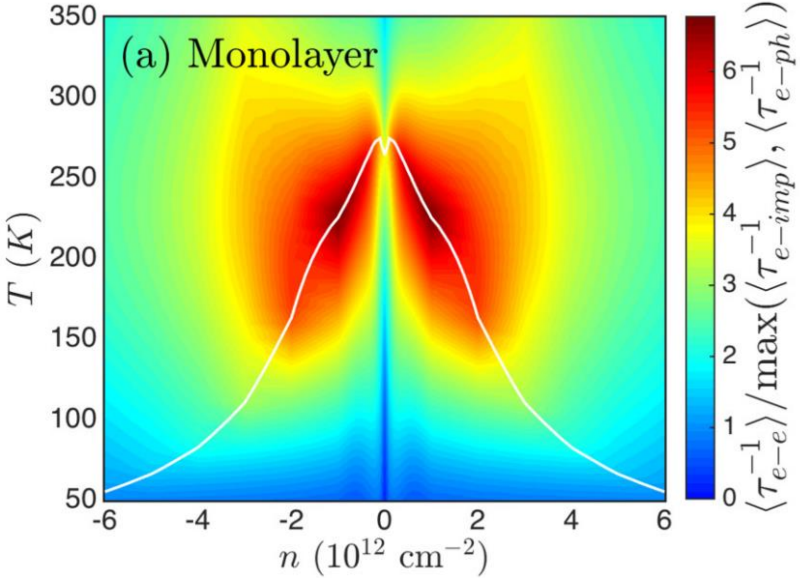
Appendix - Gurzhi Effect





Adapted from: Gurzhi, R. N., 1968 Sov. Phys. Usp. 11 255
Ho, D. Y. H. et al., Phys. Rev. B 97, 121404(R)

Appendix - Interface Modes

The system changes, and now we can write it as:

$$(\omega - Sk)Q + iSL_{-}A = 0$$
$$(\omega + Sk)R + iSL_{+}A = 0$$
$$\omega A + i\frac{1}{2}SL_{+}Q + i\frac{1}{2}SL_{-}R = 0$$

Inviscid Case

$$L_{\pm} = \frac{\partial}{\partial y} \pm \frac{\Omega_B(y)}{S}$$

Hall-Viscous Case

$$\mathscr{L}_{\pm} = S\frac{\partial}{\partial y} \pm \Omega_B(y) \mp$$
$$\mp \eta_o(y)k^2 \pm \eta_o(y)\frac{\partial^2}{\partial y^2}$$

Appendix - What have I done?

We employ the changes of variables:

(essential for chirality)

$$q = \frac{S}{n_0} n + u \quad r = \frac{S}{n_0} n - u$$

We assume transversal modulation, and longitudinal propagation

$$\begin{pmatrix} q(x,y,t) \\ r(x,y,t) \\ v(x,y,t) \end{pmatrix} = \mathrm{Re}\left[\begin{pmatrix} Q(y) \\ R(y) \\ A(y) \end{pmatrix} e^{i(kx-\omega t)} \right]$$

Majda, A. , Courant Institute of Mathematical Sciences, 2003

Appendix - Inviscid Regime

General Modes

Solving the top two equations for A, and plugging in the third one, we obtain a Schödinger equation for the amplitude A:

$$-\frac{\partial^2 A}{\partial y^2} + \left[\frac{\Omega_B^2(y)}{S^2} + \frac{k}{\omega}\frac{\partial \Omega_B(y)}{\partial y}\right] A = \frac{(\omega^2 - S^2 k^2)}{S^2} A$$

Depending on the modulation assumed for the transition, the system will present different sets of solutions!

Appendix - Inviscid Regime

Chiral Modes

We force R=0, to ensure a unidirectional propagation, making the modes chiral

$$(\omega - Sk)Q + iSL_{-}A = 0$$

$$\boxed{L_{+}A = 0}$$

$$\omega A + iS\frac{1}{2}L_{+}Q = 0$$

Trivial solution (A=0):

- Kelvin mode

Non-trivial solution:

- Yanai mode

In the Hall-viscous regime, we will only search for chiral modes

$$(\omega - Sk)Q + iS\mathscr{L}_{-}A = 0$$
$$iS\mathscr{L}_{+}A = 0$$
$$\omega A + i\frac{1}{2}S\mathscr{L}_{+}Q = 0$$

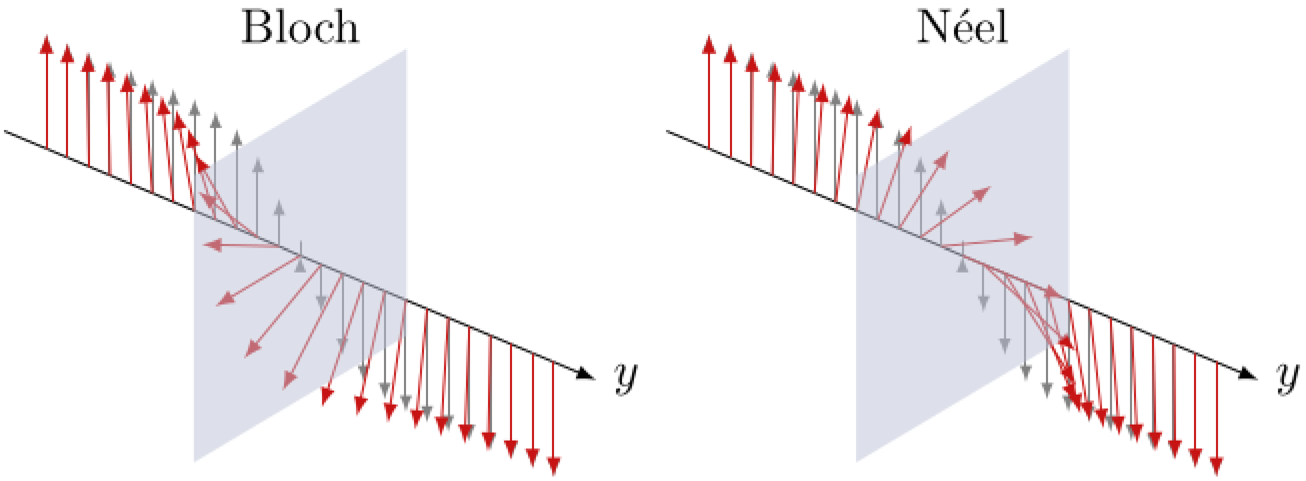
Trivial solution (A=0):
- Kelvin mode

Non-trivial solution:
- Yanai mode

$$\mathscr{L}_{\pm} = S\frac{\partial}{\partial y} \pm \Omega_B(y) \mp \eta_o(y)k^2 \pm \eta_o(y)\frac{\partial^2}{\partial y^2}$$

Appendix - Magnetic Inversions

The electrons in the 2D hydrodynamic regime are only sensitive to out-of-plane variations of the magnetic field. Therefore, we only consider the z-component of the field.



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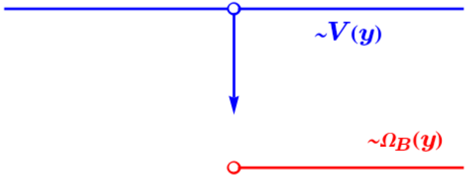
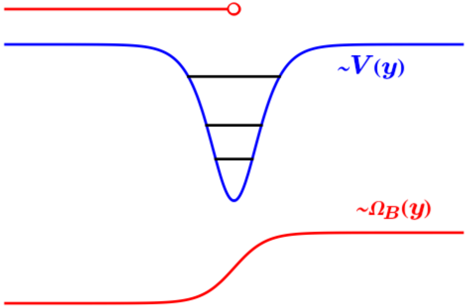
Appendix - Magnetic Inversions



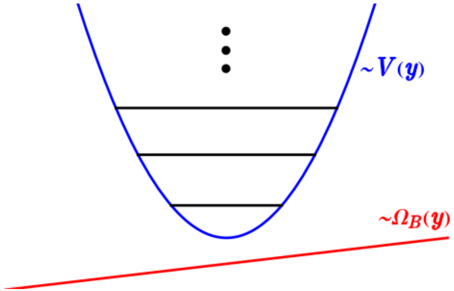
$$\Omega_B(y) = \Omega_B \,\mathrm{sgn}(y)$$

$$-\frac{\partial^2 A}{\partial y^2} + \frac{k\Omega_B}{\omega}\delta(y)A = \frac{(\omega^2 - S^2k^2 - \Omega_B^2)}{S^2}A$$



$$\Omega_B(y) = \Omega_B \tanh\left(\frac{y}{\epsilon}\right)$$

$$-\frac{\partial^2 A}{\partial y^2} - \left[\frac{k\Omega_B}{\epsilon\omega} + \frac{\Omega_B^2}{S^2}\right]\mathrm{sech}^2\left(\frac{y}{\epsilon}\right)A = \frac{(\omega^2 - S^2k^2 - \Omega_B^2)}{S^2}A$$



$$\Omega_B(y) = \frac{\Omega_B}{\epsilon}y$$

$$-\frac{\partial^2 A}{\partial y^2} - \left(\frac{\Omega_B y}{S\epsilon}\right)^2 A = \left[\frac{\omega^2}{S^2} - k^2 - \frac{\Omega_B k}{\epsilon\omega}\right]A$$

Appendix - Topology of Magnetoplasmons

We map the problem as an eigenvalue problem

$$\Psi = \begin{pmatrix} n(x,y,t) \\ u(x,y,t) \\ v(x,y,t) \end{pmatrix} = \begin{pmatrix} N \\ U \\ V \end{pmatrix} e^{i(k_x x + k_y y - \omega t)}$$

$$\omega\Psi = \hat{\mathcal{H}}\Psi \qquad \hat{\mathcal{H}} = \begin{bmatrix} 0 & ik_x & ik_y \\ ik_x & 0 & -(\Omega_B - \eta_o k^2) \\ ik_y & (\Omega_B - \eta_o k^2) & 0 \end{bmatrix}$$

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Appendix - Eigenstates

$$\sigma = \eta_o(y)/\Omega_B(y)$$

General Eigenstates:

$$\Psi_0 = \begin{bmatrix} \Omega_B(1-\sigma k) \\ -ik_y \\ ik_x \end{bmatrix} \quad , \quad \Psi_\pm = \begin{bmatrix} k/|\omega_\pm| \\ \pm\frac{k_x}{k} \pm \frac{i\Omega_B k_y}{k|\omega_\pm|}(1-\sigma k^2) \\ \pm\frac{k_y}{k} \mp \frac{i\Omega_B k_x}{k|\omega_\pm|}(1-\sigma k^2) \end{bmatrix}$$

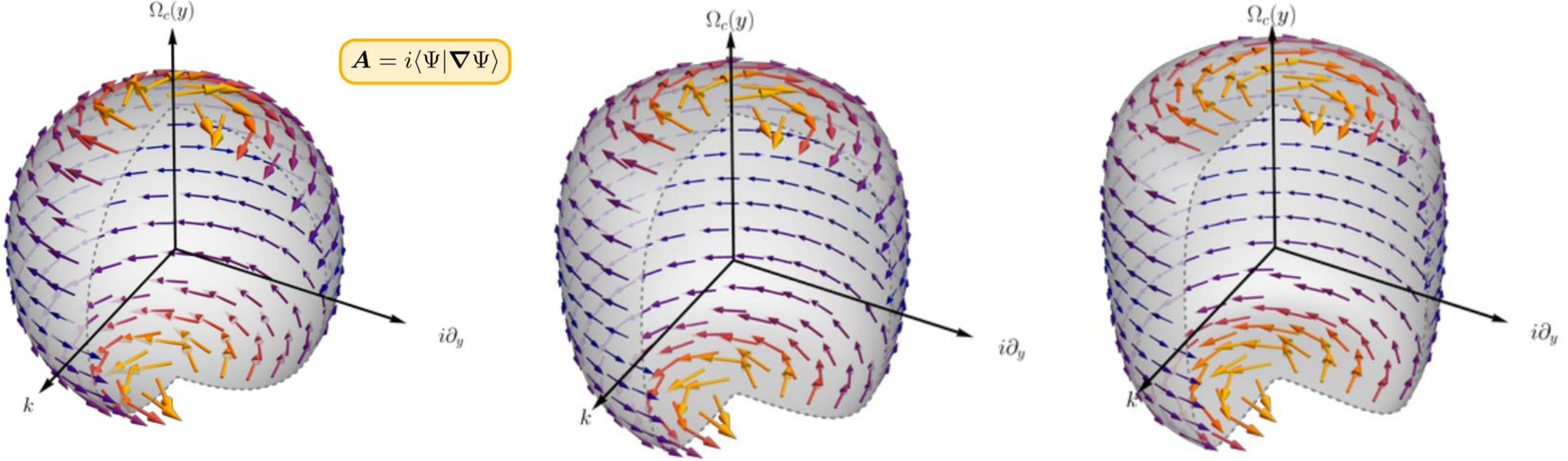
Poles in the inviscid case:

$$\Psi_+(\theta = 0) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ e^{i\phi} \\ ie^{i\phi} \end{bmatrix} \qquad \Psi_+(\theta = \pi) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ e^{-i\phi} \\ -ie^{-i\phi} \end{bmatrix}$$

Poles in the Hall-viscous case:

$$\lim_{k\to 0} \Psi_+ = \begin{bmatrix} 0 \\ \cos(\phi) - \frac{i\Omega_B \sin(\phi)}{|\Omega_B|} \\ \sin(\phi) + \frac{i\Omega_B \cos(\phi)}{|\Omega_B|} \end{bmatrix} \qquad \lim_{k\to\infty} \Psi_+ = \begin{bmatrix} 0 \\ \cos(\phi) + \frac{i\Omega_B\sigma \sin(\phi)}{|\Omega_B\sigma|} \\ \sin(\phi) - \frac{i\Omega_B\sigma \cos(\phi)}{|\Omega_B\sigma|} \end{bmatrix}$$

Appendix - Berry Curvature



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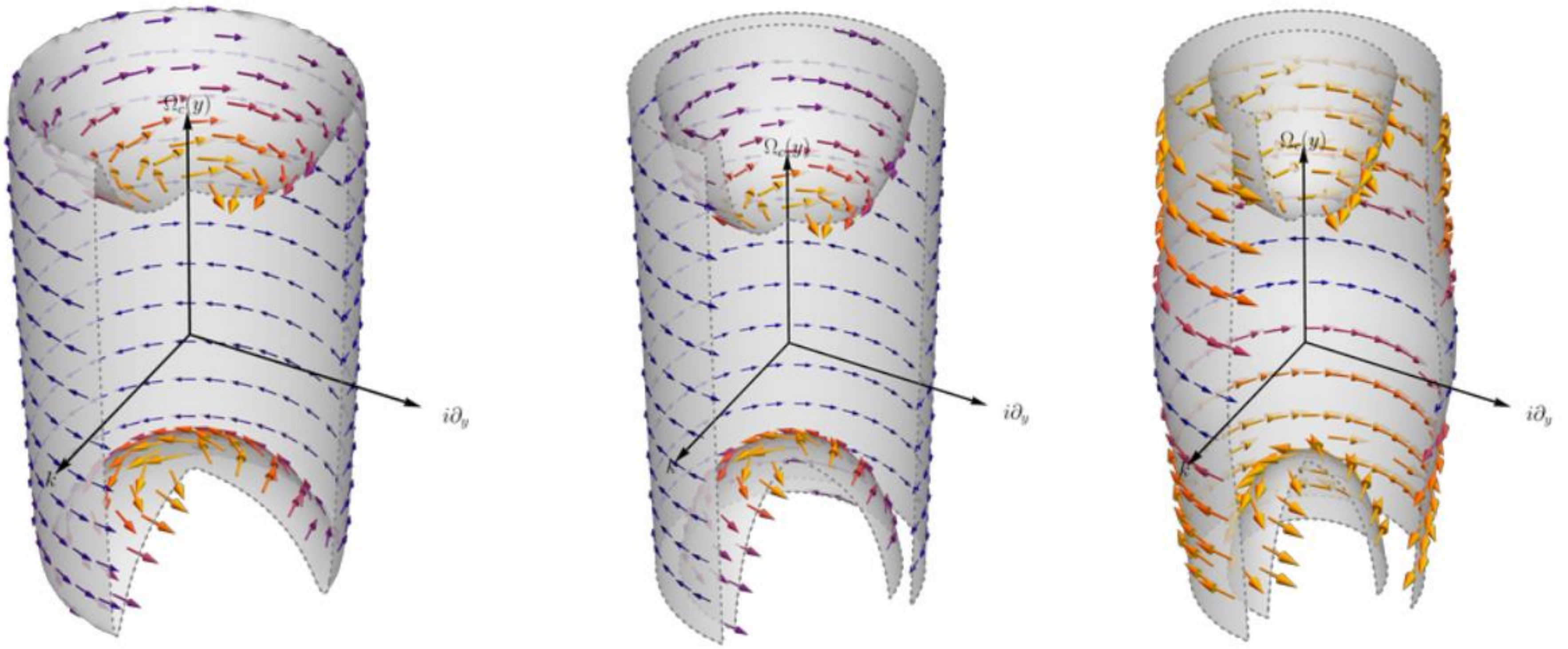
Appendix - Berry Curvatures



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Profile:

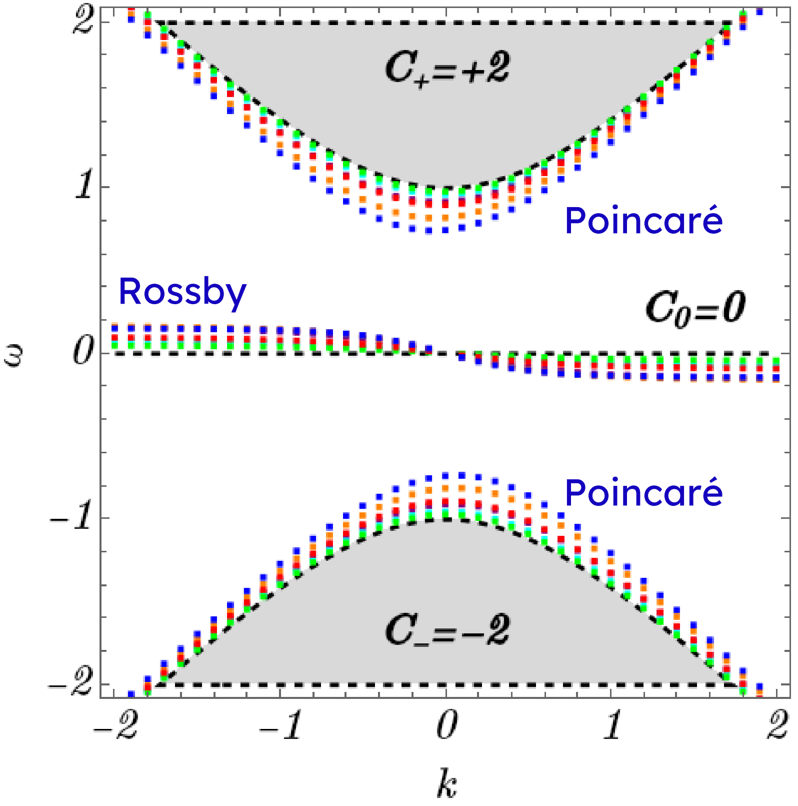
$$A(y) = P_l^{\mu}\left(\tanh\left(\frac{y}{\epsilon}\right)\right)$$

$$\ell = \sqrt{\frac{1}{4} + \Omega\epsilon\left(\frac{\Omega\epsilon}{S^2} - \frac{k}{\omega}\right)} - \frac{1}{2}$$

$$\mu = \frac{\epsilon\sqrt{k^2S^2 + \Omega^2 - \omega^2}}{S}$$

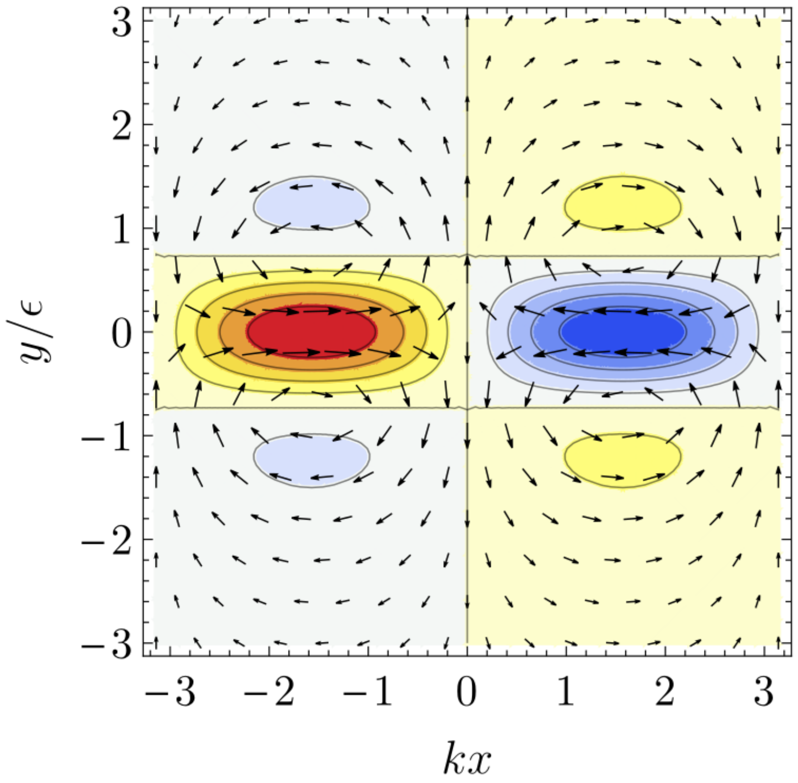
Dispersion relation:

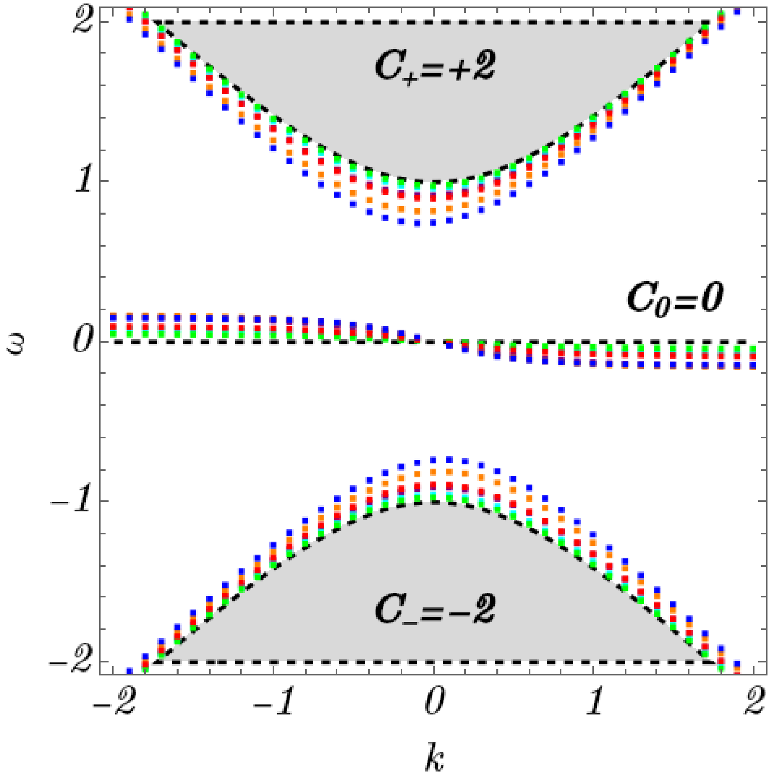
$$\ell(\ell+1)Sk^2 + \frac{\mu\Omega\sqrt{k^2S^2 - \omega^2 + \Omega^2}}{\omega}k + \frac{\Omega^2\left(\ell^2 - \mu^2 + \ell\right) - \ell(\ell+1)\omega^2}{S} = 0$$



Polyanin, A. D., Zaitsev, V. F., (2004) Handbook of Nonlinear Partial Differential Equations,Chapman and Hall/CRC

Appendix - Rossby Modes





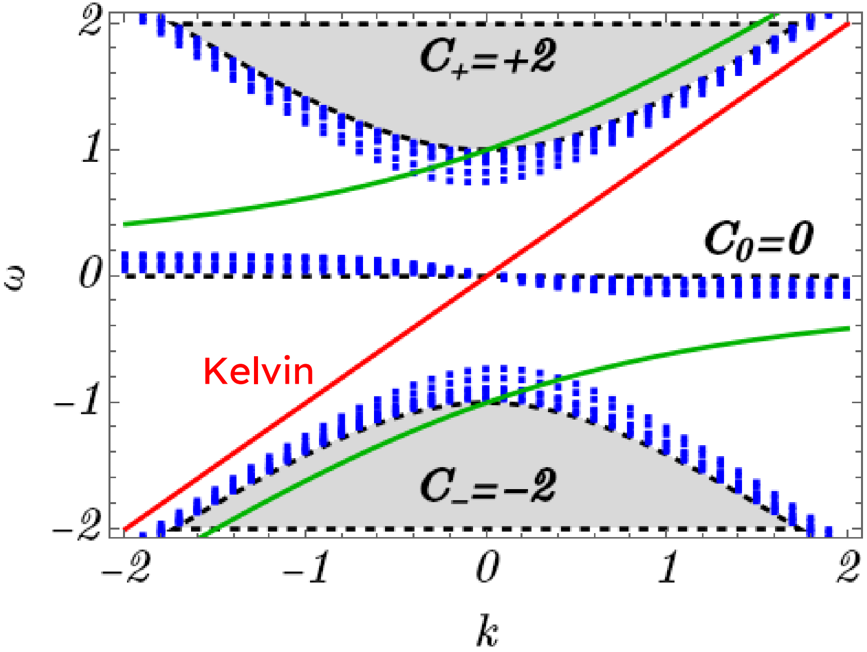
Cosme, P., Terças, H., & Santos, V. (2023), in proceedings of 2023 IEEE NMCD Conference

Profile:

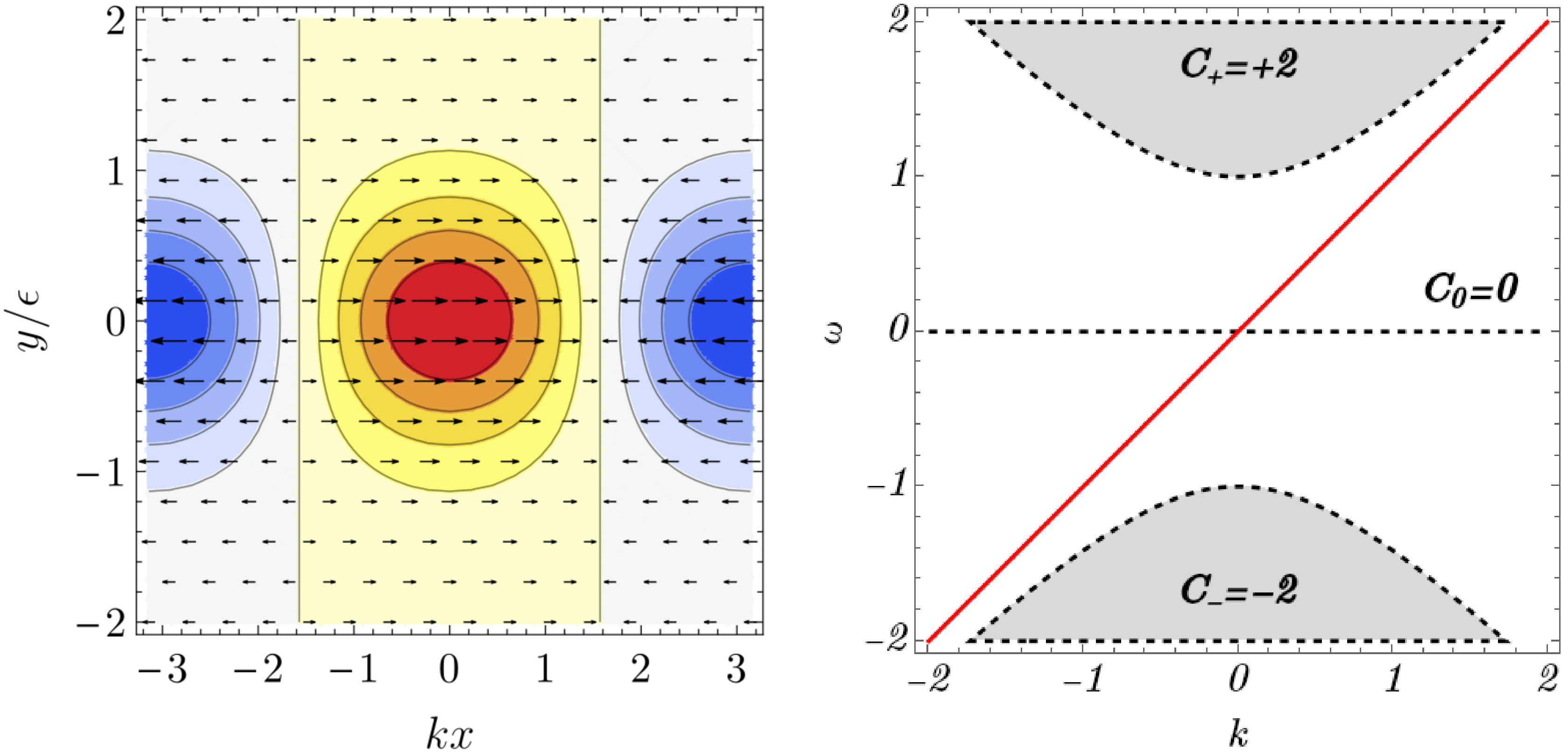
$$Q(y) = Q_0 \operatorname{sech}^{\alpha}\left(\frac{y}{\epsilon}\right) \text{ with } \alpha = \frac{\Omega_B \epsilon}{S}$$

$$A(y) = 0$$

Dispersion relation: $\omega = Sk$



Appendix - Kelvin Mode

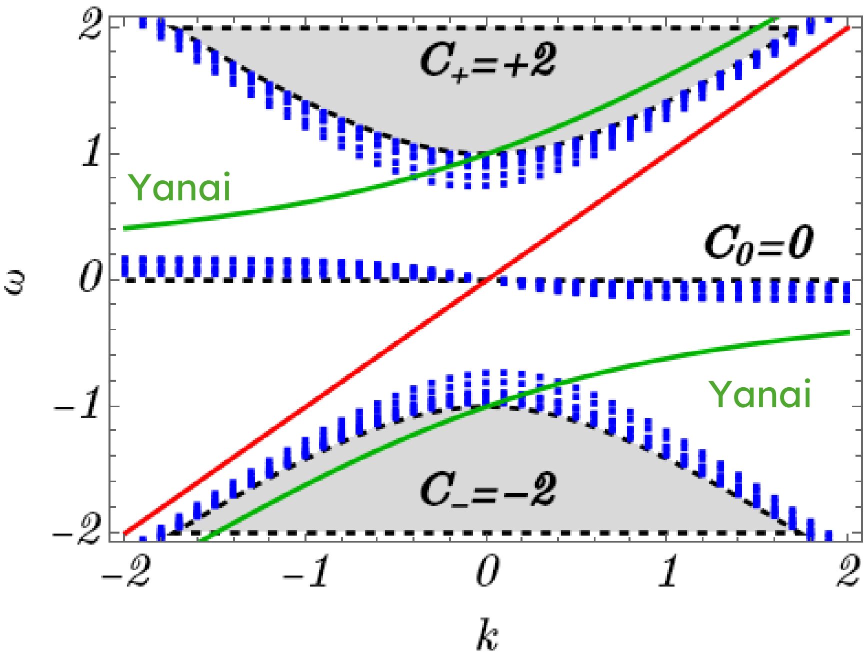


Cosme, P., Terças, H., & Santos, V. (2023), in proceedings of 2023 IEEE NMCD Conference

Profile:

$$A(y) = A_0 \,\mathrm{sech}^{\alpha}\left(\frac{y}{\epsilon}\right) \;\text{ with } \alpha = \frac{\Omega_B \epsilon}{S}$$

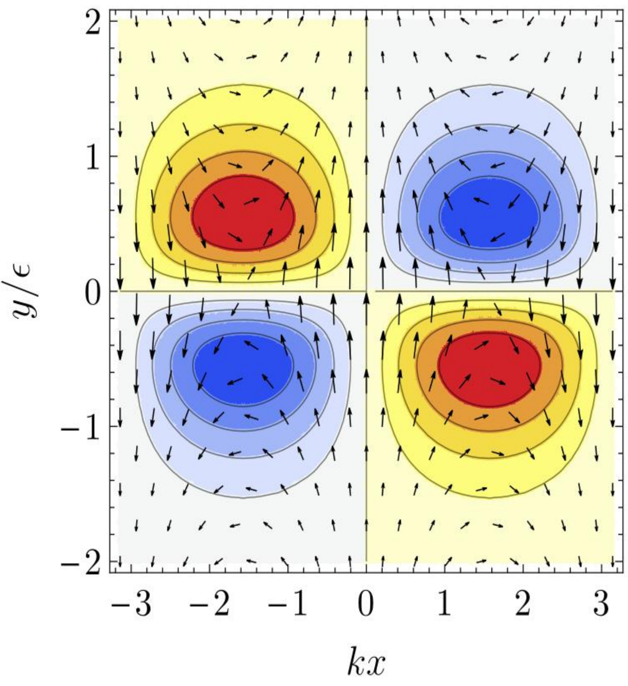
$$Q(y) = i\frac{2A_0\Omega_B}{\omega - Sk}\,\mathrm{sech}^{\alpha}\left(\frac{y}{\epsilon}\right)\tanh\left(\frac{y}{\epsilon}\right)$$

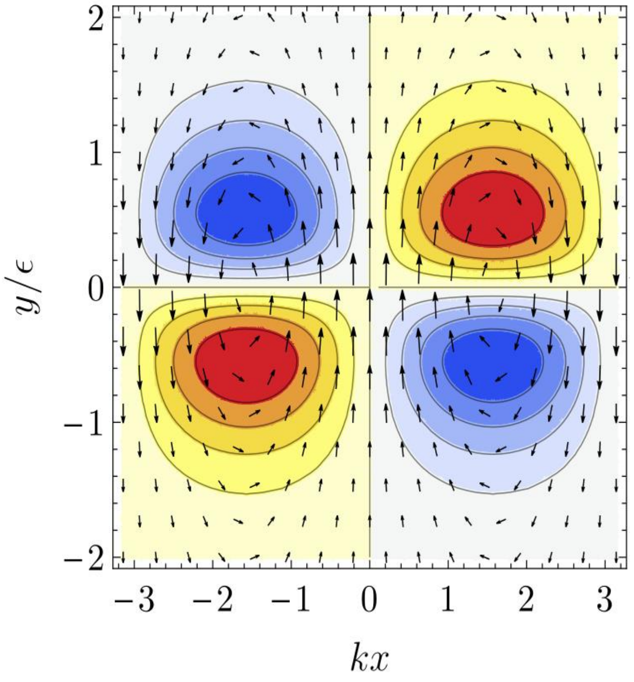


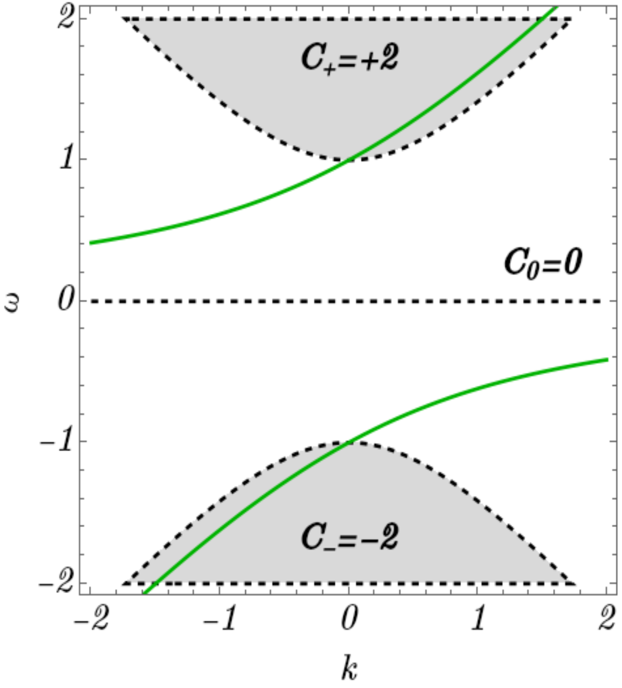
Dispersion relation:

$$\omega_{\pm} = \frac{Sk}{2} \pm \frac{S}{2}\sqrt{k^2 + 4\frac{\Omega_B}{S\epsilon}}$$

Appendix - Yanai Mode





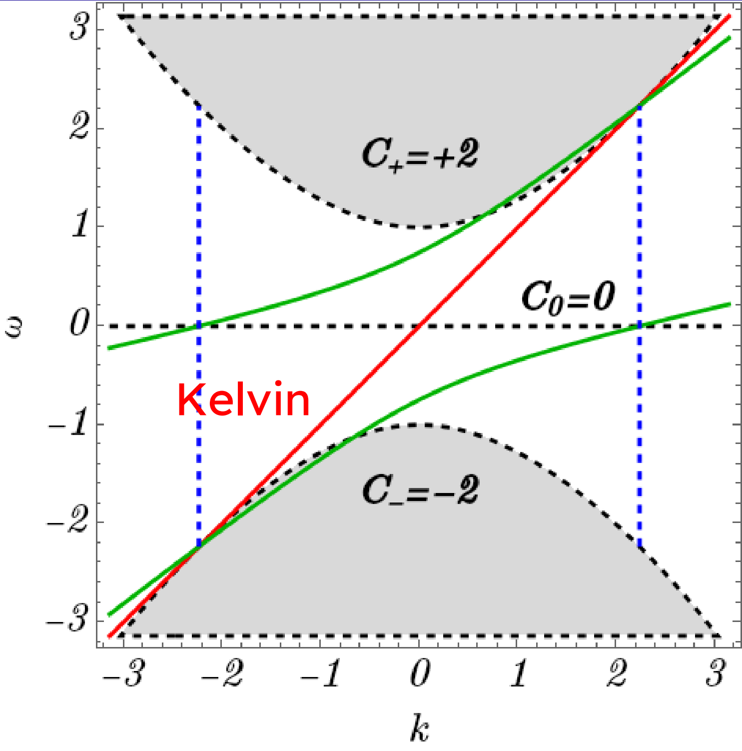


Cosme, P., Terças, H., & Santos, V. (2023), in proceedings of 2023 IEEE NMCD Conference

Profile:

$$Q(z) = (-z)^{\lambda\epsilon/2}\,{}_2F_1\left(\begin{matrix}\frac{\epsilon\lambda}{2}, \frac{\epsilon\lambda}{2}+\frac{1}{2} \\ \frac{S\epsilon}{2\eta_o}+\frac{1}{2}\end{matrix}; 1-z\right)$$

$$A(y) = 0$$

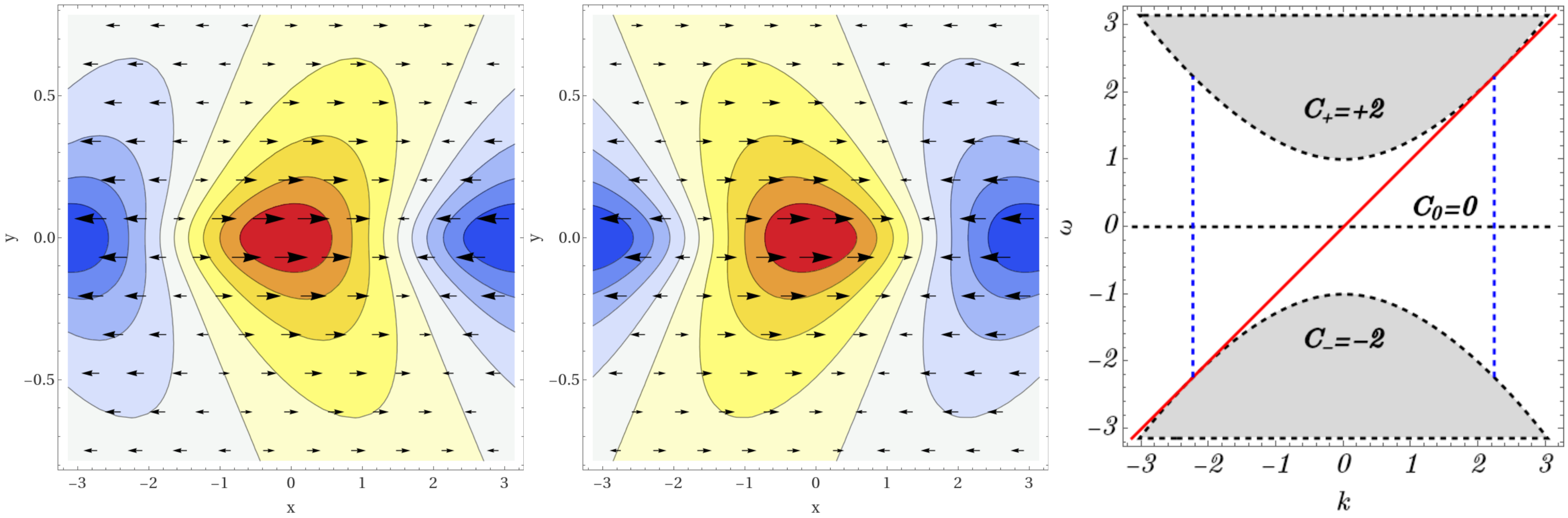


Dispersion relation: $$\omega = Sk$$

Note: λ comes from: $$\eta_o\lambda^2 - \lambda S - \eta_o k^2 + \Omega_B = 0$$

Polyanin, A. D., Zaitsev, V. F., (2004) Handbook of Nonlinear Partial Differential Equations,Chapman and Hall/CRC

Appendix - Viscid Kelvin Modes



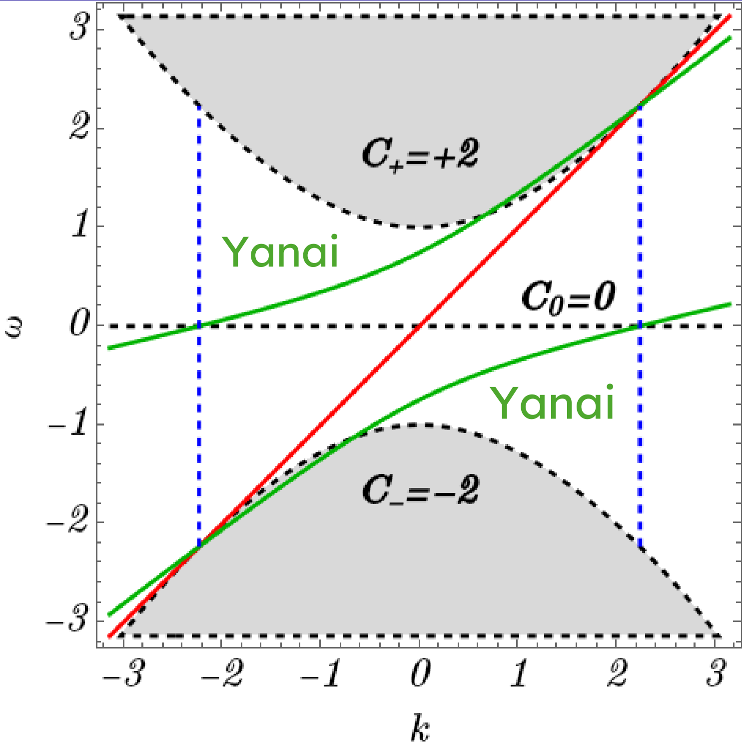
Cosme, P., Terças, H., & Santos, V. (2023), in proceedings of 2023 IEEE NMCD Conference

Appendix - Hall-Viscous Regime

Chiral Modes - Yanai Mode

Profile:

$$A(y) = (-z)^{\lambda\epsilon/2} {}_2F_1\left(\begin{matrix}\frac{\epsilon\lambda}{2},\ \frac{\epsilon\lambda}{2}+\frac{1}{2} \\ \frac{S\epsilon}{2\nu_o}+\frac{1}{2}\end{matrix};1-z\right)$$



Dispersion relation:

$$\omega_\pm = \frac{Sk}{2} \pm \frac{S}{2}\sqrt{k^2 + 2W(k)}$$

$$W(k) = \frac{2\Omega_B}{S\epsilon} - \frac{2\eta_o\left(k^2\epsilon(\eta_o + S\epsilon) + \lambda(\eta_o + 2S\epsilon - \eta_o\lambda\epsilon)\right)}{S\epsilon^2(\eta_o + S\epsilon)}$$

Note: λ comes from: $$\eta_o\lambda^2 - \lambda S - \eta_o k^2 + \Omega_B = 0$$

Polyanin, A. D., Zaitsev, V. F., (2004) Handbook of Nonlinear Partial Differential Equations,Chapman and Hall/CRC

We consider the change of variables to z, and the ansatz w(z)

$$z = \text{sech}^2(y/\epsilon) \qquad Q(z) = (-z)^{\lambda\epsilon/2} w(z)$$

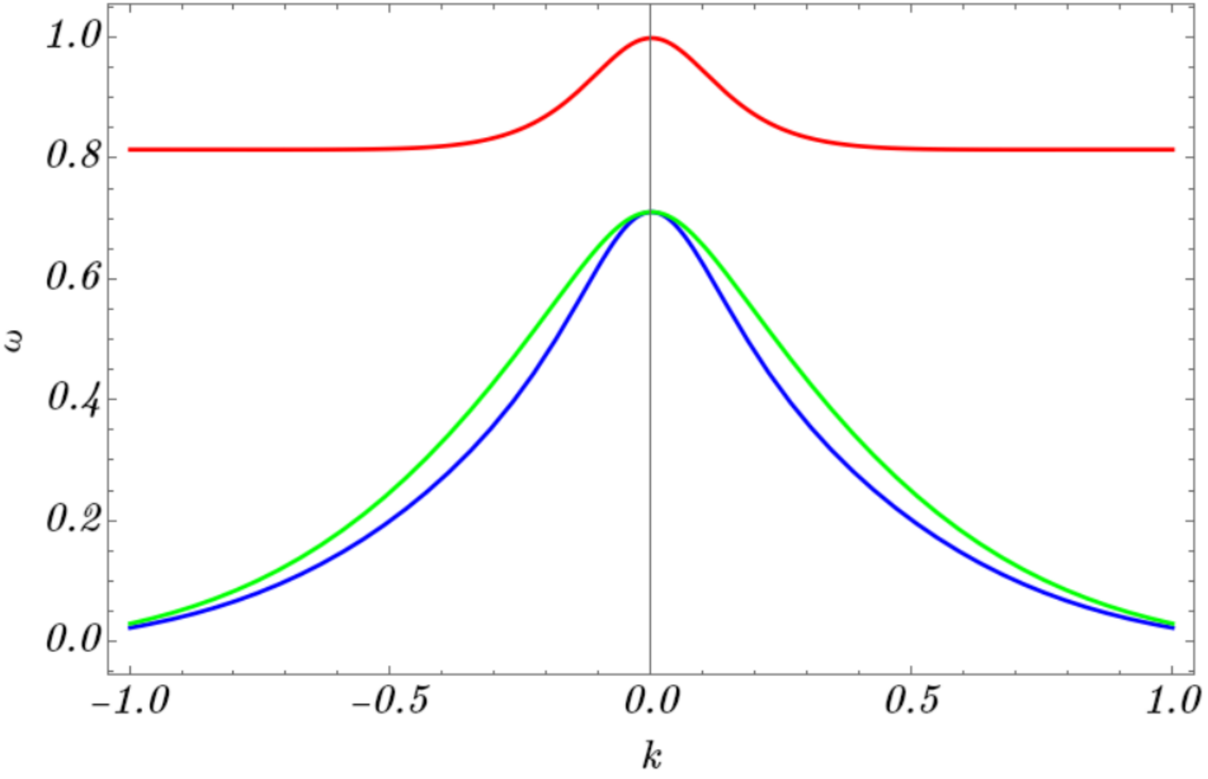
$$\eta_o \lambda^2 - \lambda S - \eta_o k^2 + \Omega_B = 0$$

$$z(1-z)w''(z) - \frac{\lambda\epsilon}{2}\cdot\frac{\lambda\epsilon+1}{2}w(z) + \left[1+\lambda\epsilon - \frac{S\epsilon}{2\eta_o} - z\left(\frac{\lambda\epsilon}{2} + \frac{\lambda\epsilon+1}{2} + 1\right)\right]w'(z) = 0$$

$$Q(z) = (-z)^{\lambda\epsilon/2}{}_2F_1\left(\begin{matrix}\frac{\epsilon\lambda}{2}, \frac{\epsilon\lambda}{2}+\frac{1}{2} \\ \frac{S\epsilon}{2\nu_o}+\frac{1}{2}\end{matrix}; 1-z\right)$$

Polyanin, A. D., Zaitsev, V. F., (2004) Handbook of Nonlinear Partial Differential Equations,Chapman and Hall/CRC
T. Morita, (1996) Interdisciplinary Information Sciences

Appendix - Ordinary Hyperbolic Function

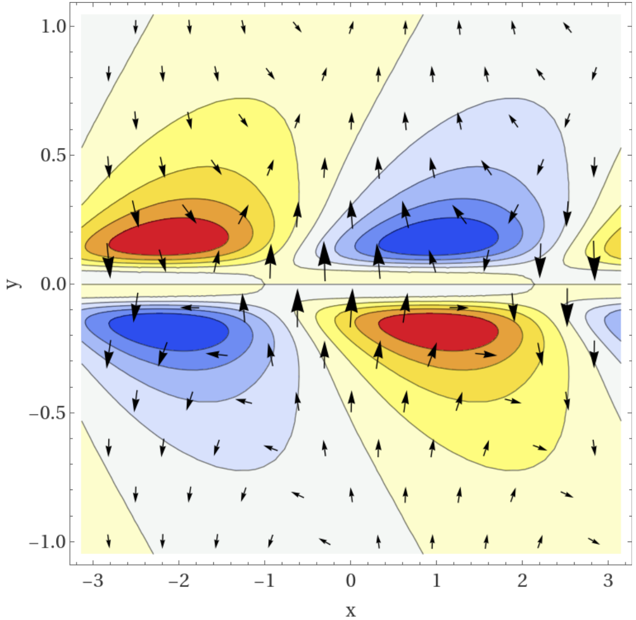


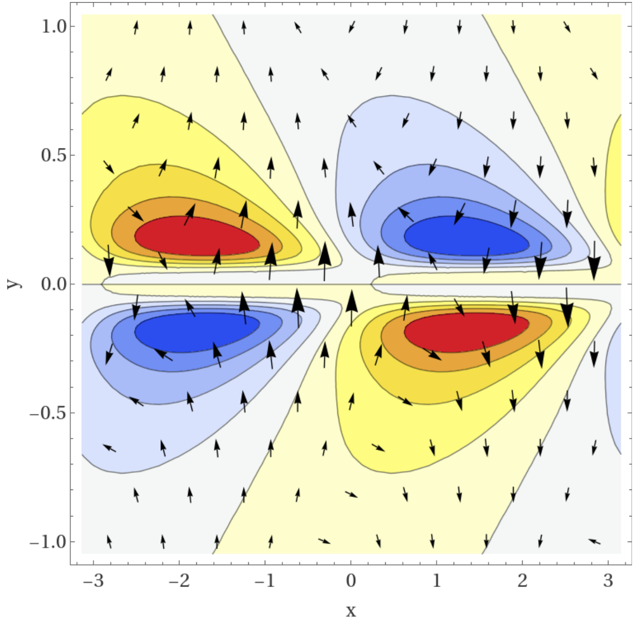
Blue: $Q(y)$;
$(-z)^{\lambda\epsilon/2}$

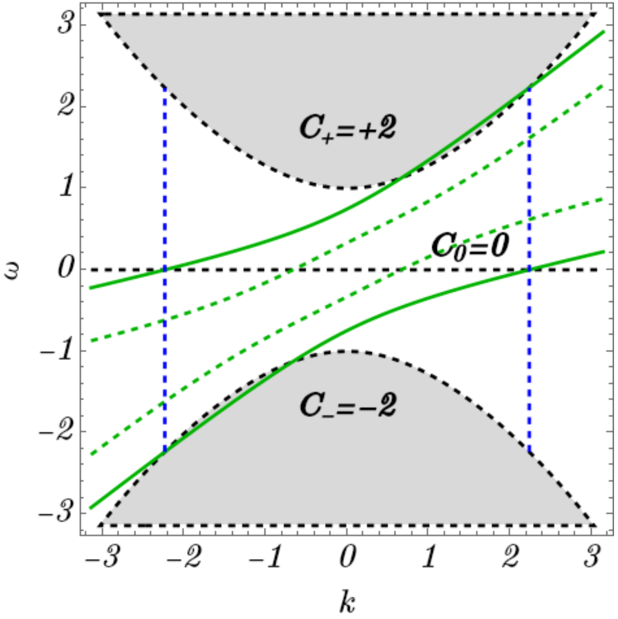
Red: ${}_2F_1(1-z)$;

Green:

Appendix - Viscid Yanai Modes







$$\omega_{\pm} = \frac{Sk}{2} \pm \frac{S}{2}\sqrt{k^2 + 2W(k)}$$

$$W(k) = \frac{2\Omega_B}{S\epsilon} - \frac{2\eta_o\left(k^2\epsilon(\eta_o + S\epsilon) + \lambda(\eta_o + 2S\epsilon - \eta_o\lambda\epsilon)\right)}{S\epsilon^2(\eta_o + S\epsilon)}$$

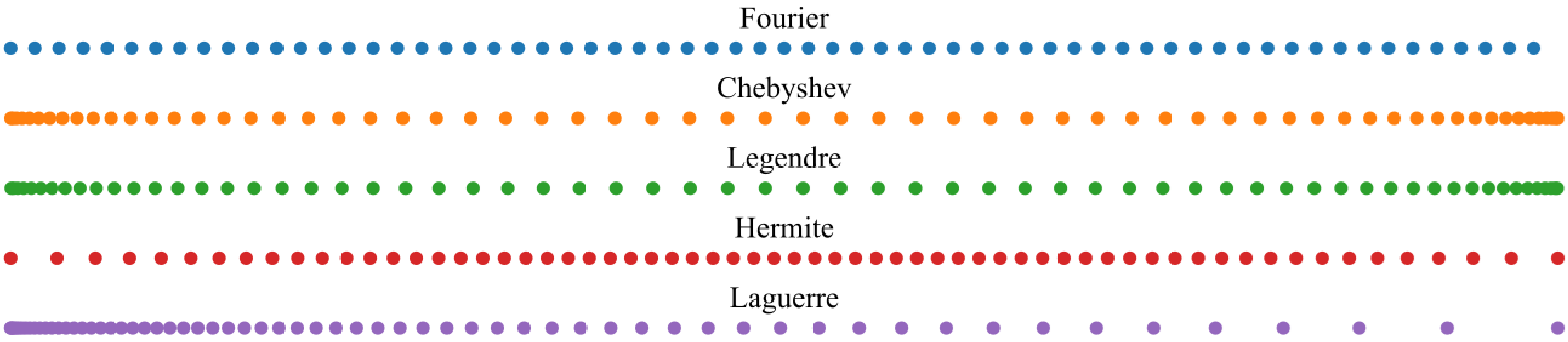
$$\eta_o\lambda^2 - \lambda S - \eta_o k^2 + \Omega_B = 0$$

Cosme, P., Terças, H., & Santos, V. (2023), in proceedings of 2023 IEEE NMCD Conference

Appendix - DEDALUS Code

DEDALUS is a library that uses spectral methods:

$$a(x) = \sum_{k=0}^{N} a_k \phi_k(x)$$



Burns, K. J. , Vasil, G. M., Oishi, J. S., Lecoanet, D., Brown, B. P., Phys. Rev. Res. (2020)

The problem is projected as an eigenvalue problem for ω.

$$\omega M \cdot \Psi + L \cdot \Psi = 0$$

$$- i\omega n + iku + J\frac{dv}{dz} + \text{lift}(\tau_1) = 0$$

$$- i\omega u + ikn - \alpha z v - \beta\left(-k^2 v + J\frac{d^2 v}{dz^2}\right) = 0$$

$$- i\omega v + J\frac{dn}{dz} + \alpha z u + \beta\left(-k^2 u + J\frac{d^2 u}{dz^2}\right) + \text{lift}(\tau_2) = 0$$

$$D_u - J\frac{du}{dz} + \text{lift}(\tau_3) = 0$$

$$D_v - J\frac{dv}{dz} + \text{lift}(\tau_4) = 0$$

$$n(z = \pm 1) = 0$$

$$\frac{dn}{dz}(z = \pm 1) = 0$$

$$\begin{pmatrix} n(x,z,t) \\ u(x,z,t) \\ v(x,z,t) \end{pmatrix} = \begin{pmatrix} N(z) \\ U(z) \\ V(z) \end{pmatrix} e^{i(kx - \omega t)}$$

$$\frac{d}{dt} = -i\omega \quad , \quad \frac{d}{dx} = ik$$

$$J = dz/dy$$

$$z \in [-1, 1]$$

$$\alpha = \frac{\Omega_B \epsilon}{S} \qquad \beta = \frac{\eta_o \Omega_B}{S^2}$$

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Appendix - Tau Method

The residual of the calculations is fitted by the program as a polynomial function with a coefficient τ.

$$\mathcal{L}(x_i) + \tau P(x_i) = f(x_i)$$

The lift operator functions as a n-order “anti-derivative”.

Tau variables are employed to force boundary conditions, if necessary.

Burns, K. J. , Vasil, G. M., Oishi, J. S., Lecoanet, D., Brown, B. P., Phys. Rev. Res. (2020)

Appendix - Mode Simulation

This is set as an initial value problem, where we provide the initial configuration:

$$M \cdot \partial_t \Psi + L \cdot \Psi = F(\Psi, t)$$

$$\Psi(x,y,0) = \begin{pmatrix} n(x,y,0) \\ u(x,y,0) \\ v(x,y,0) \end{pmatrix} = \mathrm{Re}\left[\begin{pmatrix} N(y) \\ U(y) \\ V(y) \end{pmatrix} e^{ikx} \right]$$

$$\frac{\partial n}{\partial t} + n_0 \frac{\partial u}{\partial x} + n_0 \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + \frac{S^2}{n_0} \frac{\partial n}{\partial x} - \Omega_B v - \eta_o \nabla^2 v = 0$$

$$\frac{\partial v}{\partial t} + \frac{S^2}{n_0} \frac{\partial n}{\partial y} + \Omega_B u + \eta_o \nabla^2 u = 0$$

$$x \in [-L, L]$$

$$k = n\pi / L$$

Burns, K. J. , Vasil, G. M., Oishi, J. S., Lecoanet, D., Brown, B. P., Phys. Rev. Res. (2020)