Hyperfine interactions in heavy quarkonia

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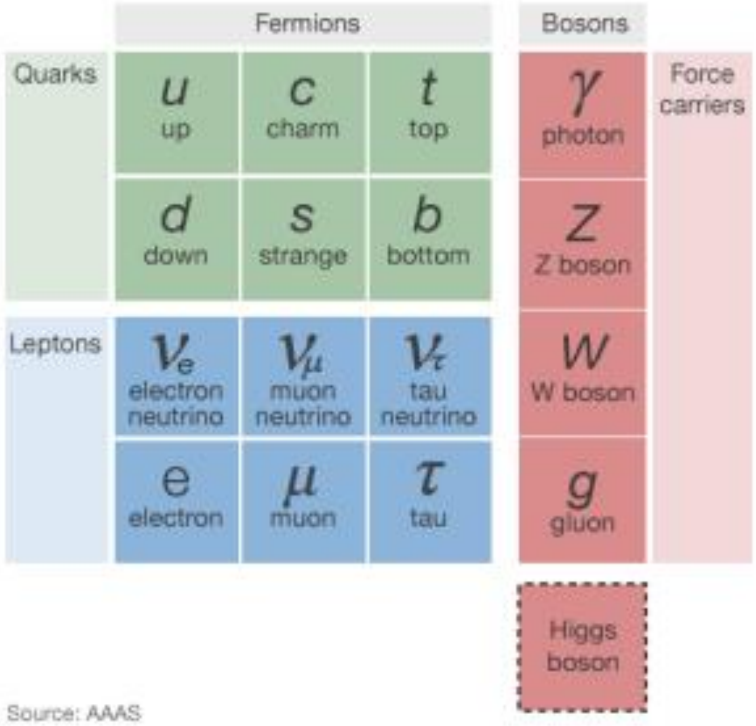
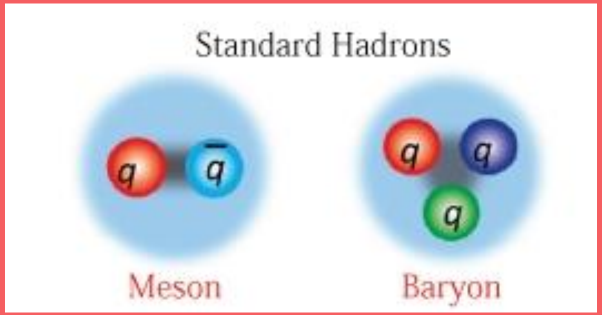
Standard Model



→ **hadrons:** bound states of strongly interacting quarks

→ **mesons:** quark-antiquark

→ **baryons:** 3 quarks

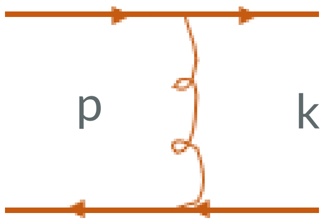


Quark and Anti-quark Potential

- One Gluon Exchange + Linear Confinement: $$V(r) = -\frac{\alpha_s}{r} + \sigma r$$
- Good description of the bottomonium spectrum

quark

anti-quark



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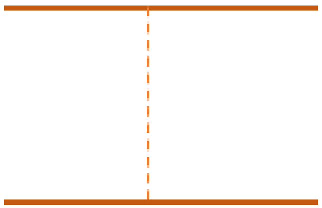


+ Relativistic terms

$$-\frac{\alpha_s}{r} \quad + \quad \sigma r$$

Objectives

- ➔ Study effects of hyperfine interaction in meson masses

  - ➔ Solve numerically the Schrödinger equation in momentum space for linear and Coulomb potentials

    Approach:

    - ➔ Treat the singularities analytically
    - ➔ Turn the S.E. into an eigen value problem

Method

➔ Obtain the Schrödinger equation in momentum space

➔ Do the partial wave decomposition for linear and Coulomb potential

$$\frac{p^2}{2m_R}\psi_\ell(p) - \frac{2\sigma}{\pi}\mathrm{P}\int_0^\infty dk\left\{\boxed{\frac{2k^2}{(k^2-p^2)^2}\left[P_\ell(y)\psi_\ell(k)-\psi_\ell(p)\right]}\right.$$

Singular at k = p

$$\left. - \frac{P_\ell'(y)}{4p^2}\boxed{\ln\left(\frac{p+k}{p-k}\right)^2}\psi_\ell(k) + \frac{2w_{\ell-1}'(y)}{4p^2}\psi_\ell(k)\right\} = E\psi_\ell(p)$$

Singular at k = p

Method

- ➔ Treat the singularities in linear and Coulomb potential
  - ➔ The idea is to rewrite our singularities into integrals we know how to calculate
  - ➔ The remaining integral will no longer be singular

$$\frac{2k^2}{(k^2-p^2)^2}\left[P_\ell(y)\psi_\ell(k)-\psi_\ell(p)\right]$$
$$=\boxed{\frac{p\psi'_\ell(p)}{k^2-p^2}}+\psi'_\ell(p)\frac{2k+p}{(k+p)^2}+\frac{2k^2R_\ell(k)}{(k+p)^2}$$

$$\longrightarrow \quad \mathrm{P}\int_0^\infty \frac{dk}{k^2-p^2}=0$$

Rewrite as

$$I_1=\int_0^\infty dk\left\{\frac{2k^2}{(k^2-p^2)^2}\left[P_\ell(y)\psi_\ell(k)-\psi_\ell(p)\right]-\frac{p\psi'_\ell(p)}{k^2-p^2}\right\}$$

Singularity free expression but with a derivative

Method

➔ Obtain singularity free S.E. for linear and Coulomb potentials

$$\left(\frac{p^2}{2m_R} - \frac{\alpha\pi p}{2} + \frac{\sigma\pi}{2p}\right)\psi_\ell(p) - \frac{\alpha}{\pi p}\int_0^\infty dk\left[kQ_\ell(y)\psi_\ell(k) - \frac{p^2}{k}Q_0(y)\psi_\ell(p)\right]$$
$$-\frac{2\sigma}{\pi}\int_0^\infty dk\left\{\frac{2k^2}{(k^2-p^2)^2}\left[P_\ell(y)\psi_\ell(k) - \psi_\ell(p)\right] - \frac{p\psi'_\ell(p)}{k^2-p^2}\right.$$
$$\left.-\frac{Q_0(y)}{2p^2}\left[P'_\ell(y)\psi_\ell(k) - \frac{p}{k}P'_\ell(1)\psi_\ell(p)\right] + \frac{W'_{\ell-1}(y)}{2p^2}\psi_\ell(k)\right\} = E\psi_\ell(p)$$

➔ Use the Gauss Legendre Quadrature to solve the integrals (discretizing p)

$$p \to p_i \quad k \to p_j$$

$$\int_0^\infty dk\, f(k) \longrightarrow \sum_{i=1}^{N} w_i f(k_i)$$

Method

➔ Evaluate this term

$$\lim_{k\to p}\left\{\frac{2k^2}{(k^2-p^2)^2}\left[P_\ell(y)\psi_\ell(k)-\psi_\ell(p)\right]-\frac{p\psi_\ell'(p)}{k^2-p^2}\right\}$$

$$=-\frac{3}{4}\psi_\ell'(p)+\frac{1}{4}\left[\frac{P_\ell'(1)}{p^2}\psi_\ell(p)+\psi_\ell''(p)\right]$$

Method

➔ Now our SE has first and second derivatives in it!

➔ We can put our derivatives in terms of an interpolating function

$$\psi_\ell(p) = \sum_j \psi_\ell(p_j) L_j(p)$$

➔ What should be this interpolation function?

Solution 1: cubic splines

➔ The derivatives are approximated to derivatives of an interpolating function

$$S_i(x) = a_i + b_i x + c_i x^2 + d_i x^3, \qquad x \in \{x_{i-1}, x_i\}$$

$$\psi_i' = \sum_j D_{ij}^{(1)} \psi_j, \qquad \psi_i'' = \sum_j D_{ij}^{(2)} \psi_j$$

➔ Choose boundary conditions to match proper wave behavior

Solution 2: Lagrange Interpolation

$$\psi_\ell(p) = \sum_j \psi_\ell(p_j) L_j(p)$$

$$L_j(p) = \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{p - p_k}{p_j - p_k} .$$

$$\psi_l'(p_i) = \sum_j L_j'(p_i) \psi_l(p_j)$$

$$\psi_l''(p_i) = \sum_j L_j''(p_i) \psi(p_j)$$

➔ Choose n p_k points to be the closest to p_i

Method

➔ Solve the eigen-value and vector problem for the matrix obtained

$$\left[\frac{p_i^2}{2\mu}+\frac{\sigma\pi}{2p_i}\frac{\ell(\ell+1)}{2}-\frac{\alpha\pi p_i}{2}\right]\Psi_\ell(p_i)-\frac{2\sigma}{\pi}\sum_{j=1,\,j\neq i}^{N} w_j\left\{\frac{2p_j^2}{(p_j^2-p_i^2)^2}[P_\ell(y_{ij})\Psi_\ell(p_j)-\Psi_\ell(p_i)]-\frac{p_i D_{ij}^{(1)}\Psi_\ell(p_i)}{p_j^2-p_i^2}\right\}$$

$$-\frac{2\sigma}{\pi}w_i\left\{\frac{3}{4p_i}D_{ij}^{(1)}\Psi_\ell(p_i)+\frac{1}{4}\left[\frac{\ell(\ell+1)}{2p_i^2}\Psi_\ell(p_i)+D_{ij}^{(2)}\Psi_\ell(p_i)\right]\right\}$$

$$+\frac{2\sigma}{\pi}\sum_{j=1,\,j\neq i}^{N} w_j\frac{1}{4p_i^2}\ln\left(\frac{p_i+p_j}{p_i-p_j}\right)^2\left[P'_\ell(y_{ij})\Psi_\ell(p_j)-P'_\ell(1)\frac{p_i}{p_j}\Psi_\ell(p_j)\right]-\frac{2\sigma}{\pi}\sum_{j=1}^{N} w_j\frac{W'_{\ell-1}(y_{ij})}{2p_i^2}\Psi_\ell(p_j)\bigg\}$$

$$-\frac{\alpha}{\pi}\sum_{j=1,\,j\neq i}^{N} w_j\frac{1}{2}\ln\left(\frac{p_i+p_j}{p_i-p_j}\right)^2\left[\frac{p_j}{p_i}P_\ell(y_{ij})\Psi_\ell(p_j)-\frac{p_i}{p_j}\Psi_\ell(p_i)\right]+\frac{\alpha}{\pi}\sum_{j=1}^{N} w_j\frac{p_j}{p_i}W_{\ell-1}(y_{ij})\Psi_\ell(p_j)=E\Psi_\ell(p_i)$$

Eigen value problem! ⟶ $$\sum_j M_{ij}\psi_j = E\psi_i$$

Results: Linear Potential with cubic splines

Test case of the Schrodinger equation with linear potential : the exact solutions for l = 0 (S wave) are known Airy Functions.
Results with 1000 integration points.

$\sigma = 1$ m_r = 1/2

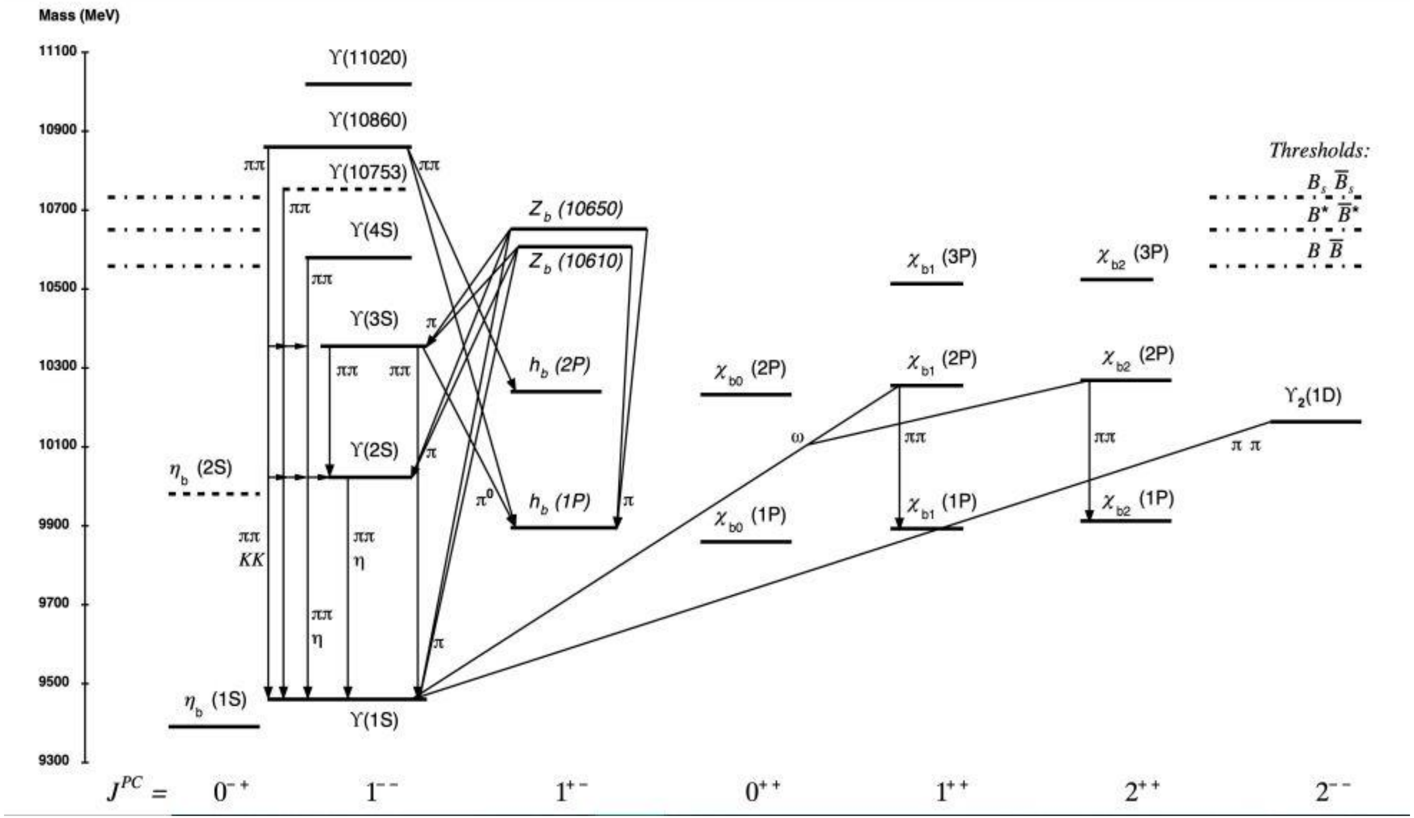
n	Numerical Results	Exact Values	n	Numerical Results	Exact Values
1	2.3381073948	2.3381074105	6	9.0226565188	9.0226508533
2	4.0879490817	4.0879494441	7	10.0401871972	10.0401743416
3	5.5205592518	5.5205598281	8	11.0085487188	11.0085243037
4	6.7867079666	6.7867080901	9	11.9360572362	11.9360155632
5	7.9441352618	7.9441335871	10	12.8288428719	12.8287767529

Results: Linear Potential with Lagrange

np = 1000 NL= 5

n	Numerical Results	Exact Values	n	Numerical Results	Exact Values
1	2.3381074105	2.3381074105	6	9.0226508400	9.0226508533
2	4.0879494442	4.0879494441	7	10.0401743111	10.0401743416
3	5.5205598281	5.5205598281	8	11.0085242419	11.0085243037
4	6.7867080889	6.7867080901	9	11.9360155581	11.9360155632
5	7.9441335823	7.9441335871	10	12.8287767217	12.8287767529

Bottomonium System



HyperFine Interactions in One Gluon Exchange

Spin-orbit

$$V(\vec{q}^{\,2}) = -\frac{4\pi\alpha_s}{\vec{q}^{\,2}}\left\{1 - \frac{\vec{k}\cdot\vec{p}}{m^2} + \boxed{\frac{3i}{4m^2}(\vec{p}\times\vec{k})\cdot(\vec{\sigma}_1+\vec{\sigma}_2)}\right.$$

$$\left.\boxed{-\frac{1}{6m^2}\vec{q}^{\,2}(\vec{\sigma}_1\cdot\vec{\sigma}_2)} + \boxed{\frac{1}{4m^2}\left[\vec{\sigma}_1\cdot\vec{q}\,\vec{\sigma}_2\cdot\vec{q} - \frac{1}{3}\vec{q}^{\,2}(\vec{\sigma}_1\cdot\vec{\sigma}_2)\right]}\right\}$$

spin-spin

Tensor Force

Pauli-Villars regularization (form factor)

$$V(\vec{q}^2) = -4\alpha_s\pi\left(\frac{1}{q^2} - \frac{1}{q^2+\Lambda^2}\right)$$

$$= \frac{4\alpha_s\pi}{q^2}\left(\frac{\Lambda^2}{q^2+\Lambda^2}\right)$$

$$\Lambda = 2m_b$$

Results with Lagrange Interpolation Bottomonium

np = 100 NL= 5 units: GeV

	n	Non-relativistic (without form factor)	Non-relativistic (with form factor in Coulomb)	Non-relativistic (with form factor in Coulomb+Linear)	Non-relativistic (with form factor + Spin-Spin)	Experimental results
l = 0	1	9.444	9.478	9.453	9.373	9.398
	2	10.004	10.018	9.993	9.962	9.999
	3	10.335	10.345	10.321	10.297	-
l = 1	1	9.913	9.913	9.888	9.888	9.899
	2	10.252	10.253	10.228	10.227	10.259
	3	10.524	10.525	10.500	10.499	-

Results with Lagrange Interpolation Charmonium

np = 100 NL= 5 units: GeV

	n	Non-relativistic (without form factor)	Non-relativistic (with form factor in Coulomb)	Non-relativistic (with form factor in Coulomb+Linear)	Non-relativistic (with form factor + Spin-Spin)	Experimental results
l = 0	1	3.076	3.104	3.020	2.957	2.984
	2	3.662	3.679	3.595	3.556	3.638
	3	4.093	4.108	4.023	3.991	-
l = 1	1	3.488	3.489	3.403	3.400	3.525
	2	3.940	3.942	3.856	3.852	-
	3	4.321	4.323	4.237	4.233	-

Results with Lagrange Interpolation

np = 100 NL= 5 units: GeV

Bottomonium

	n	Non-relativistic (with form factor + spin-spin)	Fully Relativistic
l = 0	1	9.373	9.158
	2	9.962	9.856
	3	10.297	10.207
	4	10.566	10.47873
	5	10.800	10.71291

Charmonium

	n	Non-relativistic (with form factor + spin-spin)	Fully Relativistic
l = 0	1	2.957	2.7707
	2	3.556	3.3697
	3	3.991	3.8003
	4	4.362	4.1651
	5	4.694	4.4921

Summary and Conclusions

-Solved the schrodinger equation using a linear + Coulomb potential for bottomonium and charmonium and improved the accuracy of the numerical solutions of the the linear potential through the use of a new interpolation method.

- For pseudoscalar mesons : Included the spin-spin hyperfine interaction to the potential and we observed substantial effects which makes the masses smaller.

- We compared our results with hyperfine interaction with experimental data, which show good agreance in both the meson masses.

Next Steps

- Add all the hyperfine interactions, such as the remaining ones for the Coulomb-like potential, as well as the ones arising from the linear potential.

- Adjust parameters, such as quark mass and interaction strength, through the constants alpha and sigma.

- Perform a fit to the bottomonium and charmonium spectrum.