

Hyperfine interactions in heavy quarkonia

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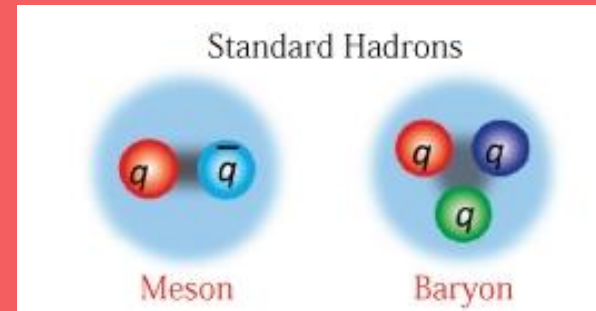
Alfred Stadler and Elmar Biernat

Standard Model

	Fermions			Bosons	
Quarks	u up	c charm	t top	γ photon	Force carriers
	d down	s strange	b bottom	Z Z boson	
Leptons	ν_e electron neutrino	ν_μ muon neutrino	ν_τ tau neutrino	W W boson	
	e electron	μ muon	τ tau	g gluon	
				Higgs boson	

Source: AAAS

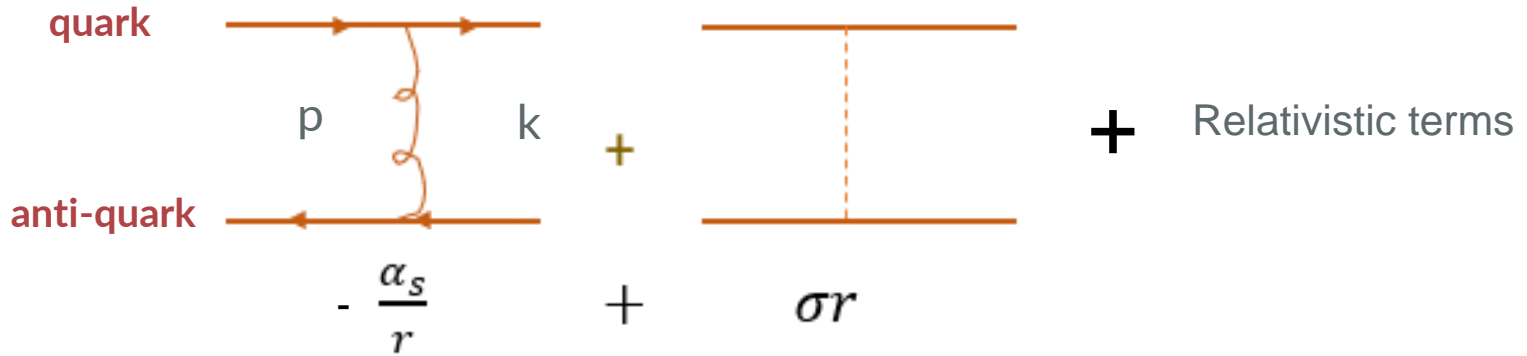
- **hadrons**: bound states of strongly interacting quarks
- **mesons**: quark-antiquark
- **baryons**: 3 quarks



Quark and Anti-quark Potential

- One Gluon Exchange + Linear Confinement:
- Good description of the bottomonium spectrum

$$V(r) = -\frac{\alpha_s}{r} + \sigma r$$



Objectives

- Study effects of hyperfine interaction in meson masses
 - Solve numerically the Schrödinger equation in momentum space for linear and Coulomb potentials

Approach:

- Treat the singularities analytically
- Turn the S.E. into an eigen value problem

Method

- Obtain the Schrödinger equation in momentum space
- Do the partial wave decomposition for linear and Coulomb potential

$$\frac{p^2}{2m_R} \psi_\ell(p) - \frac{2\sigma}{\pi} \text{P} \int_0^\infty dk \left\{ \frac{2k^2}{(k^2 - p^2)^2} [P_\ell(y) \psi_\ell(k) - \psi_\ell(p)] \right. \\ \left. - \frac{P'_\ell(y)}{4p^2} \ln \left(\frac{p+k}{p-k} \right)^2 \psi_\ell(k) + \frac{2w'_{\ell-1}(y)}{4p^2} \psi_\ell(k) \right\} = E \psi_\ell(p)$$

Singular at $k = p$

Singular at $k = p$

Method

→ Treat the singularities in linear and Coulomb potential

- The idea is to rewrite our singularities into integrals we know how to calculate
- The remaining integral will no longer be singular

$$\frac{2k^2}{(k^2 - p^2)^2} [P_\ell(y)\psi_\ell(k) - \psi_\ell(p)]$$
$$= \frac{p\psi'_\ell(p)}{k^2 - p^2} + \psi'_\ell(p) \frac{2k + p}{(k + p)^2} + \frac{2k^2 R_\ell(k)}{(k + p)^2}$$

$\longrightarrow \text{P} \int_0^\infty \frac{dk}{k^2 - p^2} = 0$

Rewrite as

$$I_1 = \int_0^\infty dk \left\{ \frac{2k^2}{(k^2 - p^2)^2} [P_\ell(y)\psi_\ell(k) - \psi_\ell(p)] - \frac{p\psi'_\ell(p)}{k^2 - p^2} \right\}$$

Singularity free expression but with a derivative

Method

→ Obtain singularity free S.E. for linear and Coulomb potentials

$$\left(\frac{p^2}{2m_R} - \frac{\alpha\pi p}{2} + \frac{\sigma\pi}{2p} \right) \psi_\ell(p) - \frac{\alpha}{\pi p} \int_0^\infty dk \left[kQ_\ell(y)\psi_\ell(k) - \frac{p^2}{k}Q_0(y)\psi_\ell(p) \right] \\ - \frac{2\sigma}{\pi} \int_0^\infty dk \left\{ \frac{2k^2}{(k^2 - p^2)^2} [P_\ell(y)\psi_\ell(k) - \psi_\ell(p)] - \frac{p\psi'_\ell(p)}{k^2 - p^2} \right. \\ \left. - \frac{Q_0(y)}{2p^2} \left[P'_\ell(y)\psi_\ell(k) - \frac{p}{k}P'_\ell(1)\psi_\ell(p) \right] + \frac{W'_{\ell-1}(y)}{2p^2}\psi_\ell(k) \right\} = E\psi_\ell(p)$$

→ Use the Gauss Legendre Quadrature to solve the integrals (discretizing p)

$$p \rightarrow p_i \quad k \rightarrow p_j \quad \int_0^\infty dk f(k) \rightarrow \sum_{i=1}^N w_i f(k_i)$$

Method

→ Evaluate this term

$$\begin{aligned} \lim_{k \rightarrow p} \left\{ \frac{2k^2}{(k^2 - p^2)^2} [P_\ell(y)\psi_\ell(k) - \psi_\ell(p)] - \frac{p\psi'_\ell(p)}{k^2 - p^2} \right\} \\ = -\frac{3}{4}\psi'_\ell(p) + \frac{1}{4} \left[\frac{P'_\ell(1)}{p^2} \psi_\ell(p) + \psi''_\ell(p) \right] \end{aligned}$$

Method

→ Now our SE has first and second derivatives in it!

→ We can put our derivatives in terms of an interpolating function

$$\psi_\ell(\mathbf{p}) = \sum_j \psi_\ell(\mathbf{p}_j) L_j(\mathbf{p})$$

→ What should be this interpolation function?

Solution 1: cubic splines

→ The derivatives are approximated to derivatives of an interpolating function

$$S_i(x) = a_i + b_i x + c_i x^2 + d_i x^3, \quad x \in \{x_{i-1}, x_i\}$$

$$\psi'_i = \sum_j D_{ij}^{(1)} \psi_j, \quad \psi''_i = \sum_j D_{ij}^{(2)} \psi_j$$

→ Choose boundary conditions to match proper wave behavior

Solution 2: Lagrange Interpolation

$$\psi_\ell(p) = \sum_j \psi_\ell(p_j) L_j(p)$$

$$L_j(p) = \prod_{\substack{k=1 \\ k \neq j}}^n \frac{p - p_k}{p_j - p_k}.$$

$$\psi'_l(p_i) = \sum_j L'_j(p_i) \psi_l(p_j)$$


$$\psi''_l(p_i) = \sum_j L''_j(p_i) \psi(p_j)$$

→ Choose n p_k points to be the closest to p_i

Method

→ Solve the eigen-value and vector problem for the matrix obtained

$$\begin{aligned}
 & \left[\frac{p_i^2}{2\mu} + \frac{\sigma\pi}{2p_i} \frac{\ell(\ell+1)}{2} - \frac{\alpha\pi p_i}{2} \right] \Psi_\ell(p_i) - \frac{2\sigma}{\pi} \sum_{j=1, j \neq i}^N w_j \left\{ \frac{2p_j^2}{(p_j^2 - p_i^2)^2} [P_\ell(y_{ij}) \Psi_\ell(p_j) - \Psi_\ell(p_i)] - \frac{p_i D_{ij}^{(1)} \Psi_\ell(p_i)}{p_j^2 - p_i^2} \right\} \\
 & - \frac{2\sigma}{\pi} w_i \left\{ \frac{3}{4p_i} D_{ij}^{(1)} \Psi_\ell(p_i) + \frac{1}{4} \left[\frac{\ell(\ell+1)}{2p_i^2} \Psi_\ell(p_i) + D_{ij}^{(2)} \Psi_\ell(p_i) \right] \right\} \\
 & + \frac{2\sigma}{\pi} \sum_{j=1, j \neq i}^N w_j \frac{1}{4p_i^2} \ln \left(\frac{p_i + p_j}{p_i - p_j} \right)^2 \left[P'_\ell(y_{ij}) \Psi_\ell(p_j) - P'_\ell(1) \frac{p_i}{p_j} \Psi_\ell(p_j) \right] - \frac{2\sigma}{\pi} \sum_{j=1}^N w_j \frac{W'_{\ell-1}(y_{ij})}{2p_i^2} \Psi_\ell(p_j) \left\{ \right. \\
 & \left. - \frac{\alpha}{\pi} \sum_{j=1, j \neq i}^N w_j \frac{1}{2} \ln \left(\frac{p_i + p_j}{p_i - p_j} \right)^2 \left[\frac{p_j}{p_i} P_\ell(y_{ij}) \Psi_\ell(p_j) - \frac{p_i}{p_j} \Psi_\ell(p_i) \right] + \frac{\alpha}{\pi} \sum_{j=1}^N w_j \frac{p_j}{p_i} W_{\ell-1}(y_{ij}) \Psi_\ell(p_j) = E \Psi_\ell(p_i) \right.
 \end{aligned}$$



 Eigen value problem! \longrightarrow $\sum_j M_{ij} \psi_j = E \psi_i$

Results: Linear Potential with cubic splines

Test case of the Schrodinger equation with linear potential : the exact solutions for $l = 0$ (S wave) are known Airy Functions.

Results with 1000 integration points.

$$\sigma = 1 \quad m_r = 1/2$$

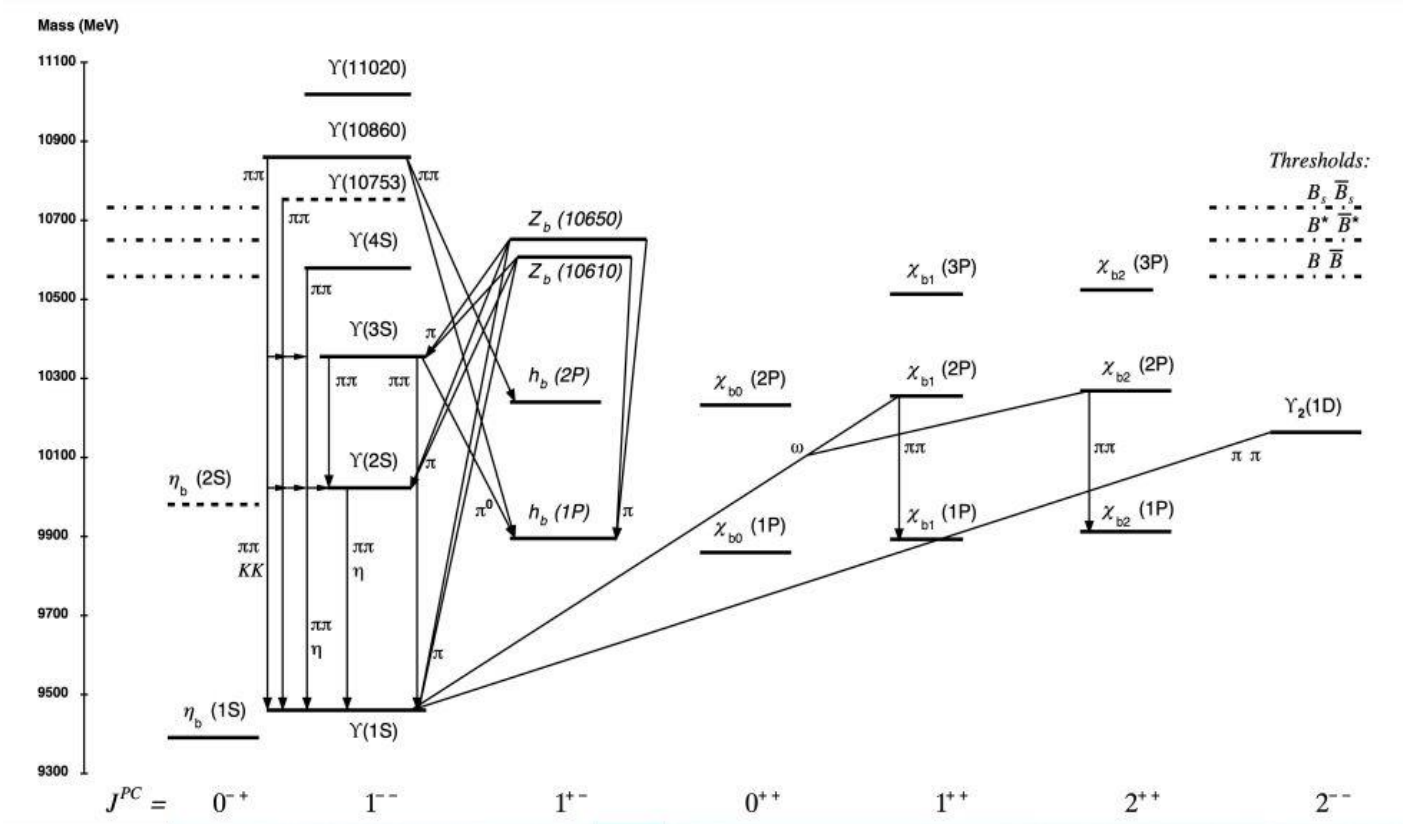
n	Numerical Results	Exact Values	n	Numerical Results	Exact Values
1	2.3381073948	2.3381074105	6	9.0226565188	9.0226508533
2	4.0879490817	4.0879494441	7	10.0401871972	10.0401743416
3	5.5205592518	5.5205598281	8	11.0085487188	11.0085243037
4	6.7867079666	6.7867080901	9	11.9360572362	11.9360155632
5	7.9441352618	7.9441335871	10	12.8288428719	12.8287767529

Results: Linear Potential with Lagrange

np = 1000 NL= 5

n	Numerical Results	Exact Values	n	Numerical Results	Exact Values
1	2.3381074105	2.3381074105	6	9.0226508400	9.0226508533
2	4.0879494442	4.0879494441	7	10.0401743111	10.0401743416
3	5.5205598281	5.5205598281	8	11.0085242419	11.0085243037
4	6.7867080889	6.7867080901	9	11.9360155581	11.9360155632
5	7.9441335823	7.9441335871	10	12.8287767217	12.8287767529

Bottomonium System



HyperFine Interactions in One Gluon Exchange

$$V(\vec{q}^2) = -\frac{4\pi\alpha_s}{\vec{q}^2} \left\{ 1 - \frac{\vec{k} \cdot \vec{p}}{m^2} + \frac{3i}{4m^2} (\vec{p} \times \vec{k}) \cdot (\vec{\sigma}_1 + \vec{\sigma}_2) \right.$$

Spin-orbit

$$\left. - \frac{1}{6m^2} \vec{q}^2 (\vec{\sigma}_1 \cdot \vec{\sigma}_2) + \frac{1}{4m^2} \left[\vec{\sigma}_1 \cdot \vec{q} \vec{\sigma}_2 \cdot \vec{q} - \frac{1}{3} \vec{q}^2 (\vec{\sigma}_1 \cdot \vec{\sigma}_2) \right] \right\}$$

spin-spin **Tensor Force**

Pauli-Villars regularization (form factor)

$$\begin{aligned} V(\vec{q}^2) &= -4\alpha_s\pi \left(\frac{1}{q^2} - \frac{1}{q^2 + \Lambda^2} \right) \\ &= \frac{4\alpha_s\pi}{q^2} \left(\frac{\Lambda^2}{q^2 + \Lambda^2} \right) \end{aligned}$$

$$\Lambda = 2m_b$$

Results with Lagrange Interpolation Bottomonium

np = 100 NL= 5 units: GeV

	n	Non-relativistic (without form factor)	Non-relativistic (with form factor in Coulomb)	Non-relativistic (with form factor in Coulomb+Linear)	Non-relativistic (with form factor + Spin-Spin)	Experimental results
l = 0	1	9.444	9.478	9.453	9.373	9.398
	2	10.004	10.018	9.993	9.962	9.999
	3	10.335	10.345	10.321	10.297	-
l = 1	1	9.913	9.913	9.888	9.888	9.899
	2	10.252	10.253	10.228	10.227	10.259
	3	10.524	10.525	10.500	10.499	-

Results with Lagrange Interpolation Charmonium

np = 100 NL= 5 units: GeV

	n	Non-relativistic (without form factor)	Non-relativistic (with form factor in Coulomb)	Non-relativistic (with form factor in Coulomb+Linear)	Non-relativistic (with form factor + Spin-Spin)	Experimental results
l = 0	1	3.076	3.104	3.020	2.957	2.984
	2	3.662	3.679	3.595	3.556	3.638
	3	4.093	4.108	4.023	3.991	-
l = 1	1	3.488	3.489	3.403	3.400	3.525
	2	3.940	3.942	3.856	3.852	-
	3	4.321	4.323	4.237	4.233	-

Results with Lagrange Interpolation

np = 100 NL= 5 units: GeV

Bottomonium

	n	Non-relativistic (with form factor + spin-spin)	Fully Relativistic
l = 0	1	9.373	9.158
	2	9.962	9.856
	3	10.297	10.207
	4	10.566	10.47873
	5	10.800	10.71291

Charmonium

	n	Non-relativistic (with form factor + spin-spin)	Fully Relativistic
l = 0	1	2.957	2.7707
	2	3.556	3.3697
	3	3.991	3.8003
	4	4.362	4.1651
	5	4.694	4.4921

Summary and Conclusions

- Solved the Schrodinger equation using a linear + Coulomb potential for bottomonium and charmonium and improved the accuracy of the numerical solutions of the linear potential through the use of a new interpolation method.
- For pseudoscalar mesons : Included the spin-spin hyperfine interaction to the potential and we observed substantial effects which makes the masses smaller.
- We compared our results with hyperfine interaction with experimental data, which show good agreement in both the meson masses.

Next Steps

- Add all the hyperfine interactions, such as the remaining ones for the Coulomb-like potential, as well as the ones arising from the linear potential.
- Adjust parameters, such as quark mass and interaction strength, through the constants α and σ .
- Perform a fit to the bottomonium and charmonium spectrum.