Quantum Entanglement and Bell inequalities with top pairs at the LHC

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Summary

- This topic [of quantum entanglement in HEP] does not seem to bring any new insight for new physics searches.
- But it is a nice twist in the interpretation of good old spin correlations in top - antitop, and more…
- And it motivates the introduction of new observables.
- It is quite fashionable now, and ATLAS and CMS are running to get the first measurements.
- Measurements will make nice headlines as the `highest-energy ever´ tests of quantum mechanics, etc.

Quantum Entanglement

Quantum entanglement

From your degree in physics, you will remember that the state of a system composed by two sub-systems A and B is separable if it can be written as

$$|\psi\rangle = |a\rangle_A \otimes |b\rangle_B$$

Otherwise, it is entangled, e.g. something like

$$|\psi\rangle = |a_1\rangle_A \otimes |b_1\rangle_B + |a_2\rangle_A \otimes |b_2\rangle_B$$

A typical example of entanglement is the combination of two spin-1/2 systems in the spin-0 configuration

$$|\psi\rangle = |\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B$$

Quantum entanglement implies that measurements on one subsystem affect the other instantaneously, even if there is a large spatial separation.

Example: top pair production

q_L anti-q_L → t anti-t gives a spin configuration $|\leftarrow\rangle \otimes |\leftarrow\rangle$ [in the q_L direction]

This is obviously not entangled.

q_R anti-q_R → t anti-t gives a spin configuration $|\rightarrow\rangle \otimes |\rightarrow\rangle$

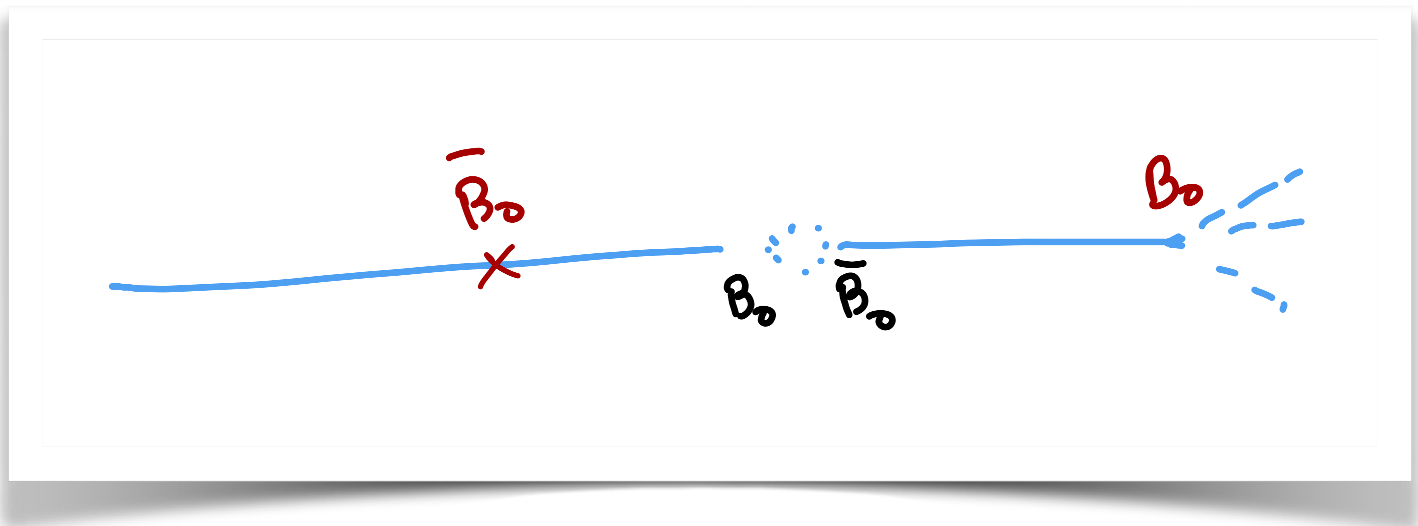
Not entangled either.

g g → t anti-t at threshold gives $\frac{1}{\sqrt{2}}\left(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle\right)$

This one is entangled.

<u>Side note</u>

Entanglement is routinely used for the measurement of time-dependent CP asymmetries in B decays, at the LHCb experiment, B factories, etc.



At the exact time one meson decays as B_0, the other one is anti-B_0

Entanglement is a genuinely quantum property of the systems.

For mixed states the definition is more complicated. But, what were mixed states?

Pure states are those that are described by a vector $|\psi\rangle$ in Hilbert space, up to a phase.

Mixed states correspond to states with classical probabilities $p_1, p_2, \ldots p_n$ for the system to be in pure states $|\psi_1\rangle, |\psi_2\rangle, \ldots |\psi_n\rangle$

They are conveniently represented by a density operator

$$\rho = p_1|\psi_1\rangle\langle\psi_1| + \cdots + p_n|\psi_n\rangle\langle\psi_n|$$

Of course, this is different from the pure state

$$p_1|\psi_1\rangle + \cdots + p_n|\psi_n\rangle$$

Example: top pair production

q anti-q → t anti-t is 50% of the time q_L anti-q_L and 50% of the time q_R anti-q_R

Then, we have 50% of the time $|\leftarrow\rangle \otimes |\leftarrow\rangle$ and 50% $|\rightarrow\rangle \otimes |\rightarrow\rangle$

Obviously, in q anti-q → t anti-t we do have t anti-t spin correlations. But not entanglement!

[Dropping anti- from now on…]

This example also illustrates the use of the density operator formalism. Otherwise, we could not describe q q → t t !

Any operator cannot be a density operator. A valid density operator has several characteristics:

- Unit trace
- Hermitian
- Positive semidefinite: eigenvalues ≥ 0

A density operator describing a composite system is separable if it can be written as

$$\rho_{\text{sep}} = \sum_n p_n \rho_n^A \otimes \rho_n^B$$

Note: in general, one has something like

$$\rho = \sum_{ijkl} p_{ij}^{kl} |\psi_i\rangle\langle\psi_j| \otimes |\psi_k\rangle\langle\psi_l|$$

Necessary criterion for separability:

Peres, quant-ph/9604005
Horodecki, quant-ph/9703004

taking the transpose in subspace of B [for example] the resulting density operator is valid.

Example: composite system A ⊗ B with dim $\mathcal{H}_A$ = n, dim $\mathcal{H}_B$ = m

P_{ij} are m x m matrices, $(P_{ij})^{kl} = p_{ij}^{kl}$

$$\rho = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & & \\ \vdots & & \ddots & \\ P_{n1} & & & P_{nn} \end{pmatrix} \quad \Rightarrow \quad \rho^{T_2} = \begin{pmatrix} P_{11}^T & P_{12}^T & \cdots & P_{1n}^T \\ P_{21}^T & P_{22}^T & & \\ \vdots & & \ddots & \\ P_{n1}^T & & & P_{nn}^T \end{pmatrix}$$

(n∗m) × (n∗m) matrix

Not easily tractable!

Quantum entanglement

To take away:

- It is quite complicated to prove [analytically] that a composite system is in a separable state.
- However, we are interested in showing that the system is entangled.
- For that, simple sufficient conditions are enough.

ρ^{T2} non-positive ⇒ ρ^{T2} not valid ⇒ system entangled

Showing this for a single vector is enough → simple conditions

Top pair production

Top quarks have spin 1/2, as it is well known.

This corresponds to a Hilbert space $\mathcal{H}$ of dimension 2

I have mentioned that a valid density operator is Hermitian and with unit trace. Therefore, I can `expand´ it in terms of Pauli matrices as

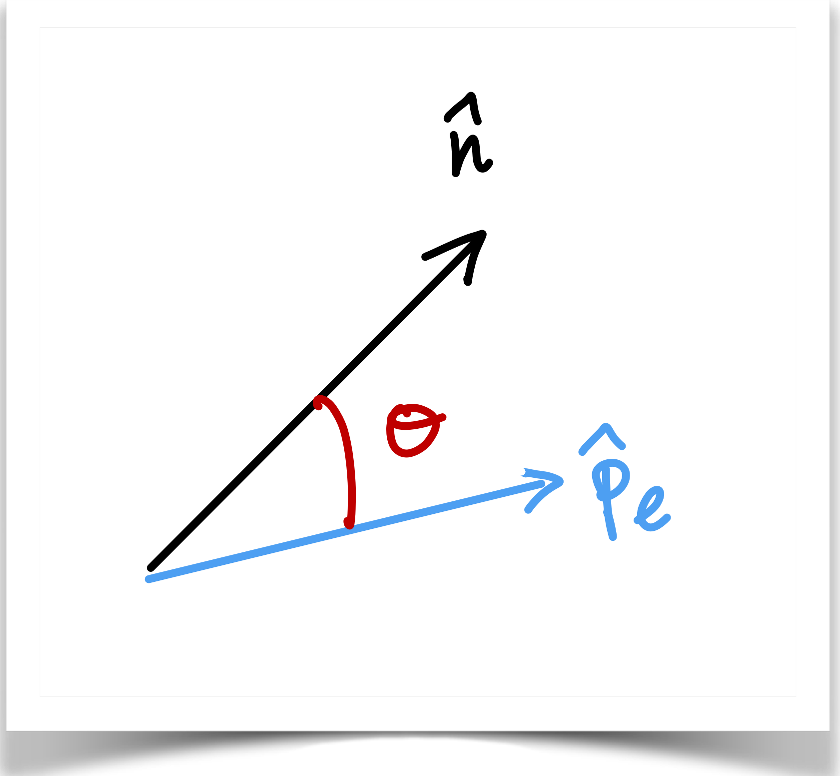
$$\frac{1}{2}\left(1_{2\times 2} + \sum_i B_i^+ \sigma_i\right) \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The B_i are constants and correspond to the top polarisation . There are additional degrees of freedom [momentum] that we can integrate out, or consider a specific region in phase space.

The spin of the top quark cannot be directly measured, but statistically the spin state can be determined from angular distributions.

The charged lepton distribution in the top quark rest frame, with respect to any axis n, is

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta} = \frac{1}{2}(1 + \alpha P_{\hat{n}} \cos\theta)$$



constant that depends on decay product: α = 1 for charged lepton

Polarisation: 2 × <S>

The charged lepton distribution allows to measure expected value of spin operators for the top quark / antiquark

When we have a top-antitop pair, we have a composite system of two spin-1/2 particles.

The `spin space´ is $\mathcal{H}_A \otimes \mathcal{H}_B$, of dimension 2 × 2.

The density operator for the top-antitop pair can be written as

$$\rho = \frac{1}{4}\left(1 \otimes 1 + \sum_i B_i^+ \sigma_i \otimes 1 + \sum_i B_i^- 1 \otimes \sigma_i + \sum_{ij} C_{ik} \sigma_i \otimes \sigma_j\right)$$

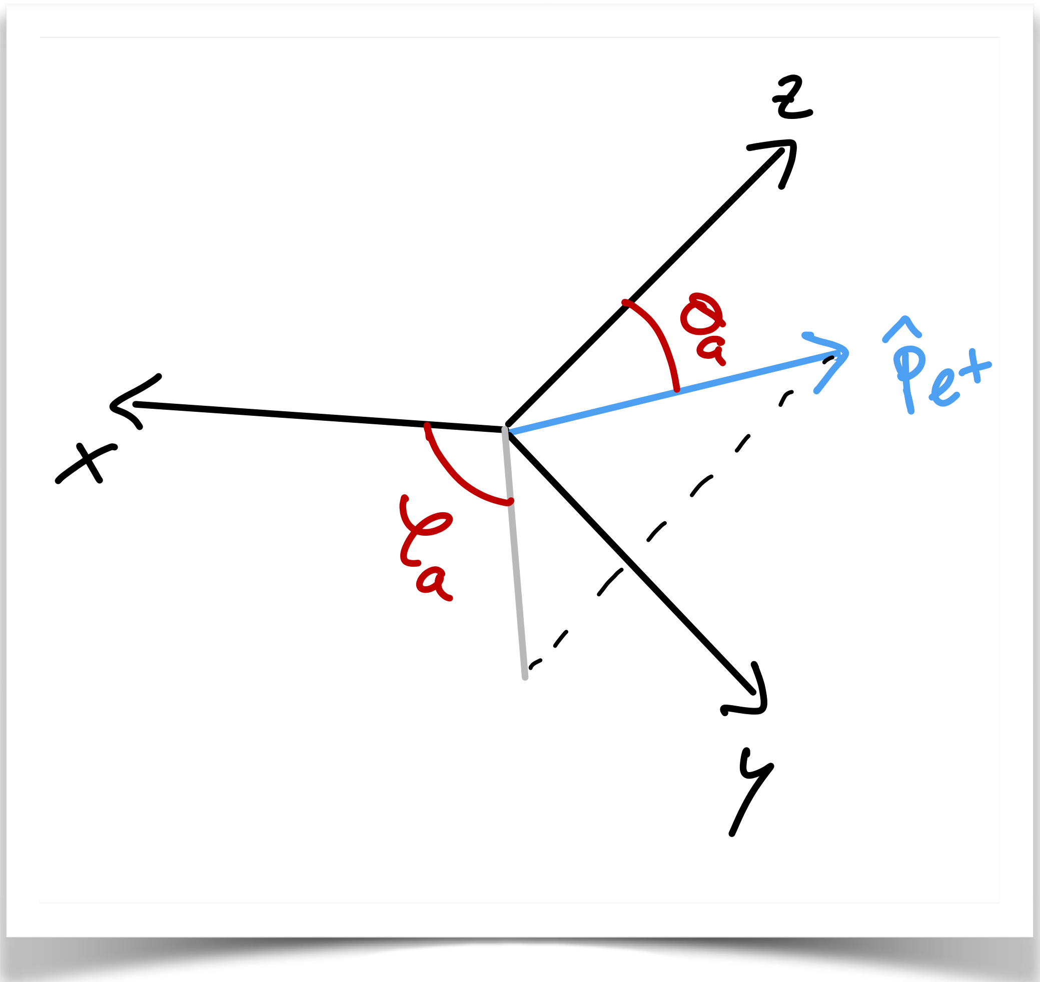
polarisation of top

polarisation of anti-top

spin correlations

The identification of the coefficients with polarisations, etc. can be done by calculating expected values of spin operators

Again, the *B* and *C* coefficients characterising the spin state of top pair production can be measured from the charged lepton distributions, fixing a reference system



ℓ^+ from top: θ_a ϕ_a

ℓ^- from anti-top: θ_b ϕ_b

The corresponding 4-dimensional distribution for the charged leptons is maybe a bit frightening

$$\frac{1}{\sigma}\frac{d\sigma}{d\Omega_a d\Omega_b} = \frac{1}{(4\pi)^2}\left[1\right.$$
$$+\alpha_a\left(B_1^+ \sin\theta_a \cos\varphi_a + B_2^+ \sin\theta_a \sin\varphi_a + B_3^+ \cos\theta_a\right)$$
$$+\alpha_b\left(B_1^- \sin\theta_b \cos\varphi_b + B_2^- \sin\theta_b \sin\varphi_b + B_3^- \cos\theta_b\right)$$
$$+\alpha_a\alpha_b \sin\theta_a \sin\theta_b \left(C_{11} \cos\varphi_a \cos\varphi_b + C_{22} \sin\varphi_a \sin\varphi_b\right)$$
$$+\alpha_a\alpha_b \sin\theta_a \sin\theta_b \left(C_{12} \cos\varphi_a \sin\varphi_b + C_{21} \sin\varphi_a \cos\varphi_b\right)$$
$$+\alpha_a\alpha_b \left(C_{13} \sin\theta_a \cos\varphi_a \cos\theta_b + C_{31} \cos\theta_a \sin\theta_b \cos\varphi_b\right)$$
$$+\alpha_a\alpha_b \left(C_{23} \sin\theta_a \sin\varphi_a \cos\theta_b + C_{32} \cos\theta_a \sin\theta_b \sin\varphi_b\right)$$
$$\left.+\alpha_a\alpha_b C_{33} \cos\theta_a \cos\theta_b\right]$$

normalisation

polarisation of top

polarisation of anti-top

spin correlations

but with suitable integrations the coefficients in red can be extracted from LHC data. And they have been.

ATLAS, 1612.07004
CMS, 1907.03729

Top pair Entanglement

I have mentioned that simple sufficient conditions for entanglement can be written.

For the case of the top quark, some of these conditions are

$$|C_{11} + C_{22}| > 1 + C_{33}$$
$$|C_{11} - C_{22}| > 1 - C_{33}$$

These remarkably simple conditions result from requiring $\langle a|\rho^{T_2} a\rangle < 0$ for strategically-chosen vectors *a*

The coefficients C_{ij} are just the ones ATLAS and CMS have measured

ATLAS, 1612.07004
CMS, 1907.03729

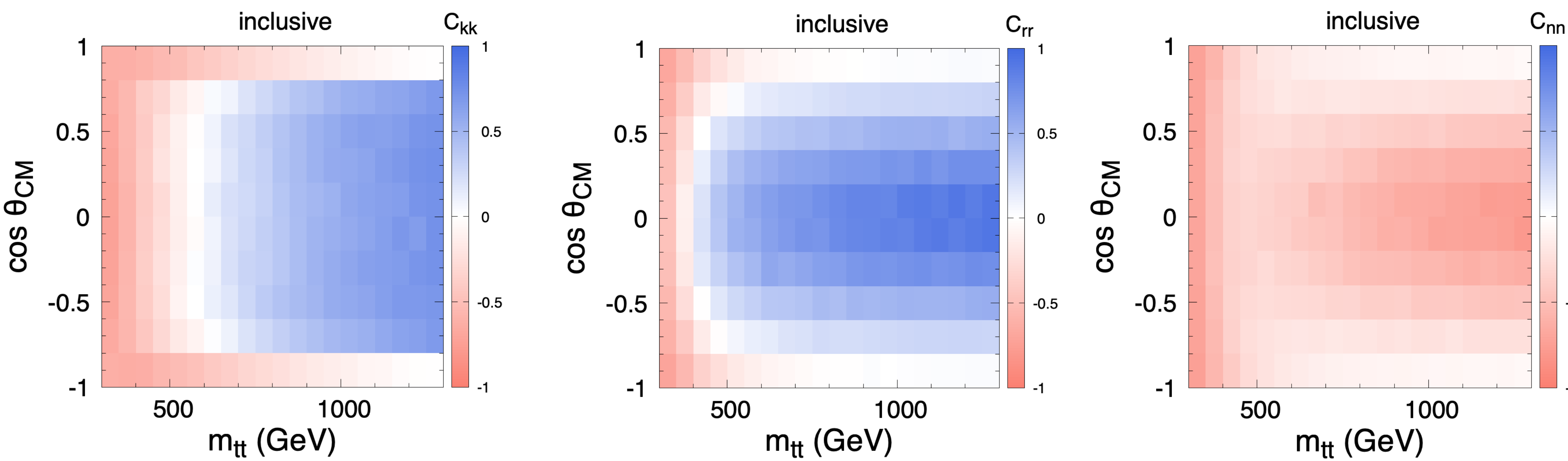
Observables already measured by ATLAS and CMS allow to test the entanglement of the top pair

Top pair entanglement

There is a dependence of the C_{ij} coefficients on the kinematics.

In the helicity basis:

Notice: $C_{nn} < 0$



K: top helicity

R: ⊥ in production plane

N: ⊥ to K and N

$$\hat{k} = \hat{p}_t$$

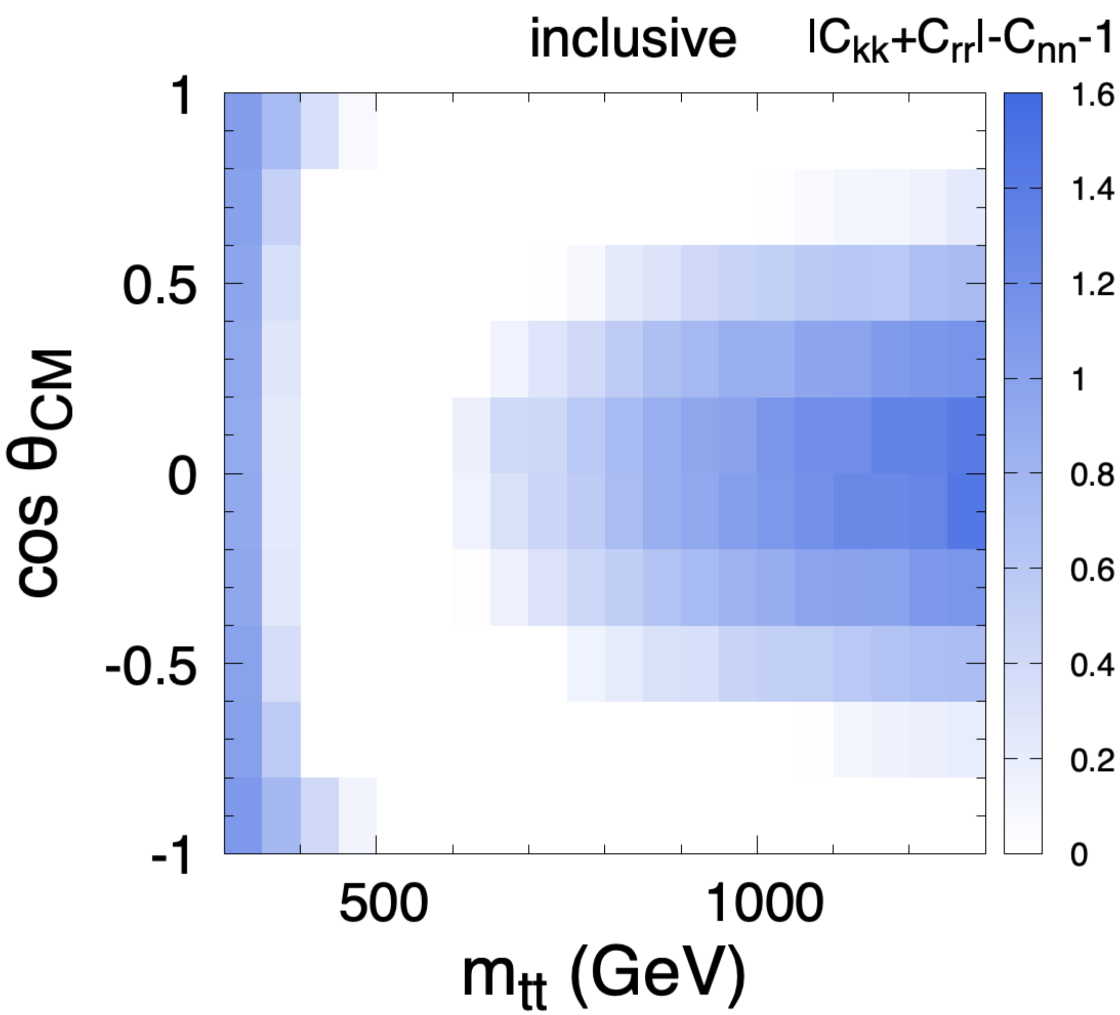
$$\hat{r} \propto [\hat{p}_p - (\hat{p}_p \cdot \hat{p}_t)\hat{k}]$$

$$\hat{n} = \hat{k} \times \hat{r}$$

1 → K axis; 2 → R axis; 3 → N axis

Since $C_{nn} < 0$, one of the sufficient conditions is stronger:

$$|C_{kk} + C_{rr}| - C_{nn} > 1$$



Near threshold:

$C_{kk} + C_{rr} < 0$

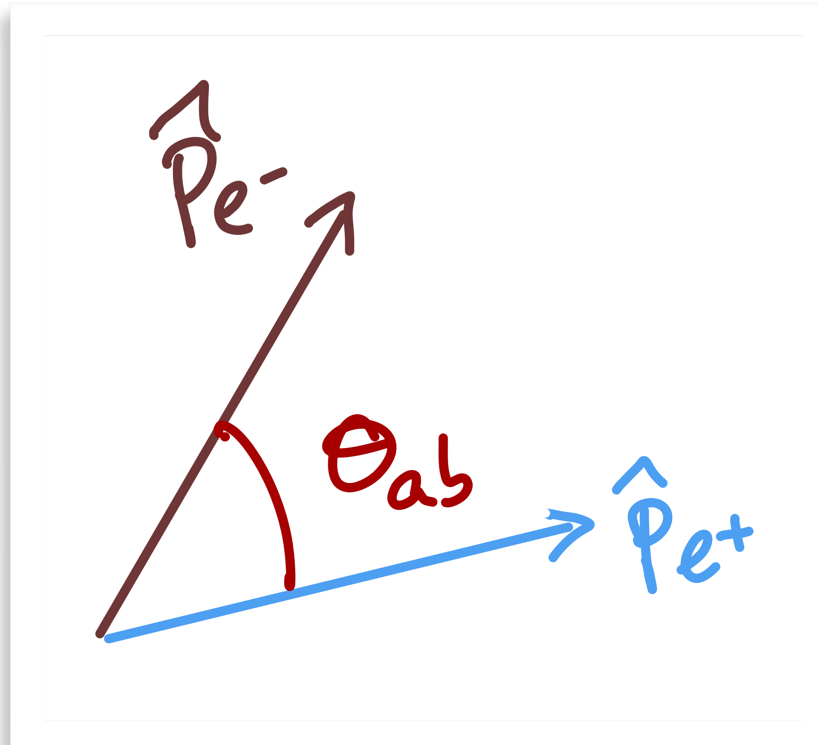
Measure $-C_{kk} - C_{rr} - C_{nn}$

Boosted, central:

$C_{kk} + C_{rr} > 0$

Measure $C_{kk} + C_{rr} - C_{nn}$

ATLAS and CMS are pursuing a measurement at threshold using the D observable, related to the angle between the two leptons

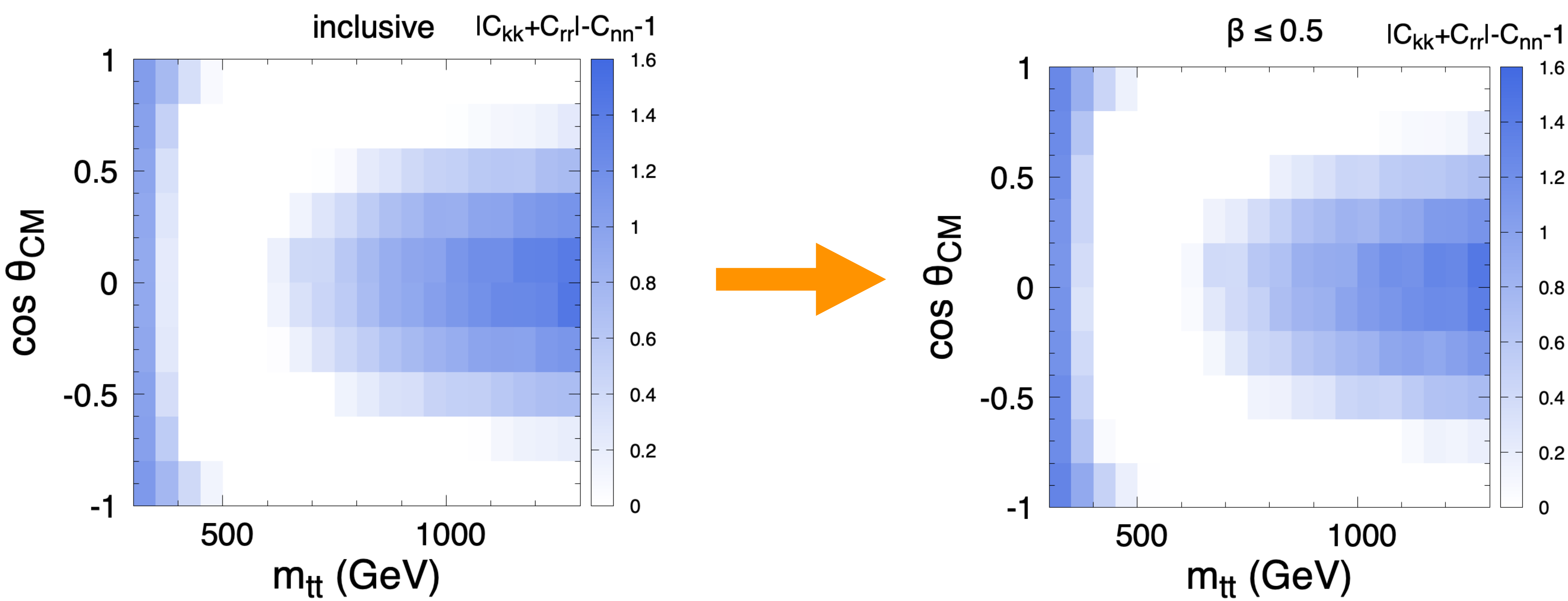


$$\frac{1}{\sigma}\frac{d\sigma}{d\cos\theta_{ab}} = \frac{1}{2}\left(1 + \alpha_a \alpha_b D \cos\theta_{ab}\right) \qquad D = \frac{1}{3}(C_{11} + C_{22} + C_{33})$$

Entanglement test at threshold: $-3D - 1 > 0$

Possible improvement: consider events that are more central: upper cut on tt velocity β in LAB frame

JAAS, Casas, 2205.00542



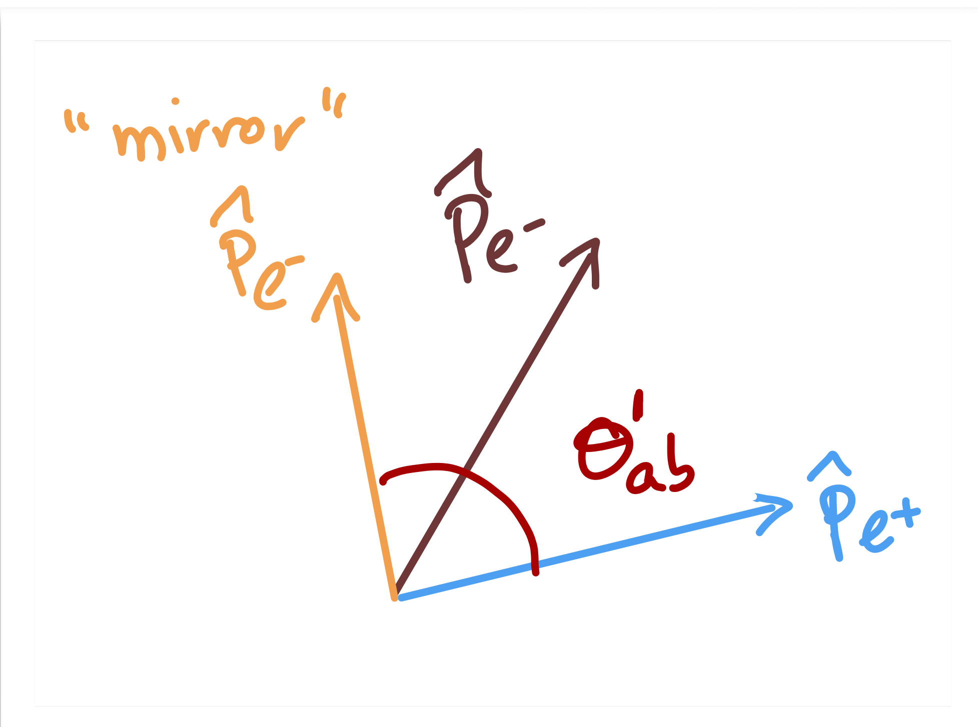
- opposite contributions from *qq* and *gg* sub-processes
- The upper cut reduces the *qq* fraction.

What about the boosted central region?

The relevant quantity to test is $C_{kk} + C_{rr} - C_{nn}$ and there is no specific observable for this combination [one can however measure C's and sum]

Let's build a new one!

JAAS, Casas,2205.00542



Use the mirror image of ℓ^- momentum, reflected in the K-R plane

$$\frac{1}{\sigma}\frac{d\sigma}{d\cos\theta'_{ab}} = \frac{1}{2}\left(1 + \alpha_a\alpha_b D_3 \cos\theta'_{ab}\right)$$

$$D_3 = \frac{1}{3}(C_{11} + C_{22} - C_{33})$$

Entanglement test for boosted region: $3D_3 - 1 > 0$

Why dedicated observable?

- If one measures C_{kk}, C_{rr} and C_{nn} independently, and performs the sum, the statistical uncertainty is larger.
- Systematic uncertainties require a detailed study.

however

Measuring C_{kk}, C_{rr} and C_{nn} requires full reconstruction of the K, R and N axes. Measuring the angle θ'_{ab} only requires to reconstruct the N axis.

How much is the improvement?

Setting systematics aside, there is an improvement of the statistical uncertainty of the `entanglement indicator´ $E = |C_{kk} + C_{rr}| - C_{nn} - 1$

LHC Run 2 139 fb^{-1}

threshold [β] $E: \quad 0.559 \pm 0.017 \longrightarrow 0.679 \pm 0.019$ 1.27 ×

boosted [D_3] $E: \quad 0.671 \pm 0.069 \longrightarrow 0.663 \pm 0.056$ 1.23 ×

Near threshold there are quite large statistics, but in the boosted central region there are not.

1.23 × improvement in statistical sensitivity for the boosted region is equivalent to 50% more luminosity!

Bell inequalities

Bell inequalities

Bell-like inequalities hold for classical systems. Their violation implies quantum mechanics.

In particular, the violation implies that the quantum system is not described by hidden variables.

Bell-like inequalities are based on measurements on two separate sub-systems A [Alice] and B [Bob], of photons, electrons, …

Experiments usually performed measuring spins

Bell inequalities

A useful formulation of Bell-like inequalities is provided by the so-called CHSH inequalities for two systems A (Alice) and B (Bob).

Clauser, Horne, Shimony, Holt, '69

Alice measures two spin observables *A*, *A*´. Bob measures two spin observables *B*, *B*´. [Both normalised to unity]. Then, clasically:

$$|\langle AB\rangle - \langle AB'\rangle + \langle A'B\rangle + \langle A'B'\rangle| \leq 2$$

these are spin correlation observables!

One can show violation of CHSH inequalities if one finds spin observables *A*, *A*´ for Alice and *B*, *B*´ for Bob such that the inequality is violated.

in a given quantum state!

Bell inequalities

The CHSH inequalities involve spin correlations. Therefore, for a particle of spin 1/2, they involve the C_{ij} spin-correlation coefficients [seen for top pair production]

It can be shown that the maximum of the l.h.s.

$$|\langle AB\rangle - \langle AB'\rangle + \langle A'B\rangle + \langle A'B'\rangle|$$

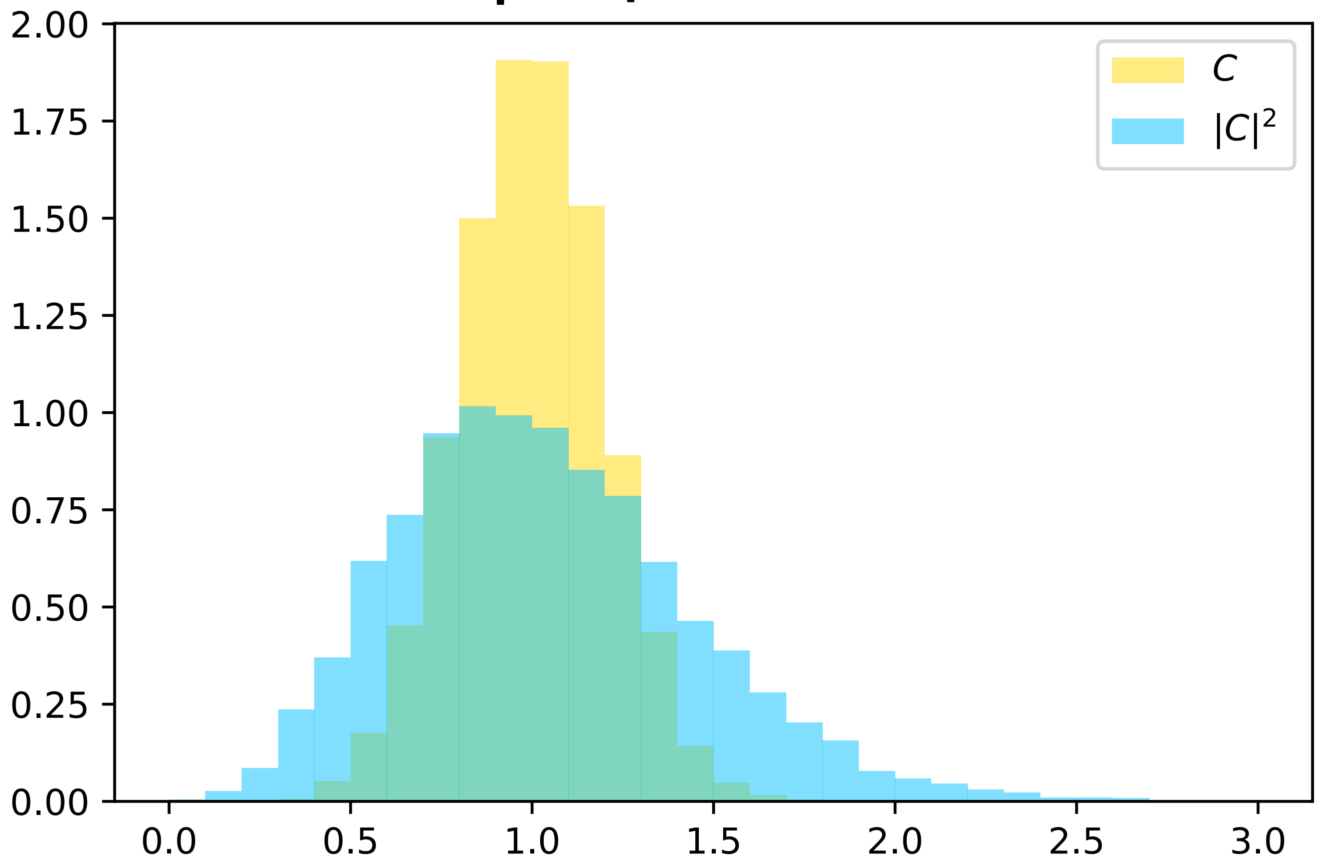
is given by

$$2\sqrt{\lambda_1 + \lambda_2}$$

where λ_1 and λ_2 are the two largest eigenvalues of the positive definite matrix C^TC Horodecki, Horodecki, Horodecki, ' 95

However, statistical fluctuations bias the eigenvalues of $C^T C$ towards larger values

Example: μ = 1, σ = 0.2



… this \`optimal´ procedure is not as good as it seems, and simpler ways are more robust and equally effective!

Severi et al. 2110.10112

Take simple choice of [non-commuting] spin observables

$$A \to 2S_i \qquad B \to \frac{1}{\sqrt{2}}(2S_i + 2S_j)$$

$$A' \to 2S_j \qquad B' \to \frac{1}{\sqrt{2}}(-2S_i + 2S_j)$$

$$|\langle AB \rangle - \langle AB' \rangle + \langle A'B \rangle + \langle A'B' \rangle| \;\to\; |C_{ii} + C_{jj}|$$

$$|\langle AB \rangle - \langle AB' \rangle + \langle A'B \rangle + \langle A'B' \rangle| \;\to\; |C_{ii} - C_{jj}|$$

$$A \to 2S_i \qquad B \to \frac{1}{\sqrt{2}}(-2S_i - 2S_j)$$

$$A' \to 2S_j \qquad B' \to \frac{1}{\sqrt{2}}(2S_i - 2S_j)$$

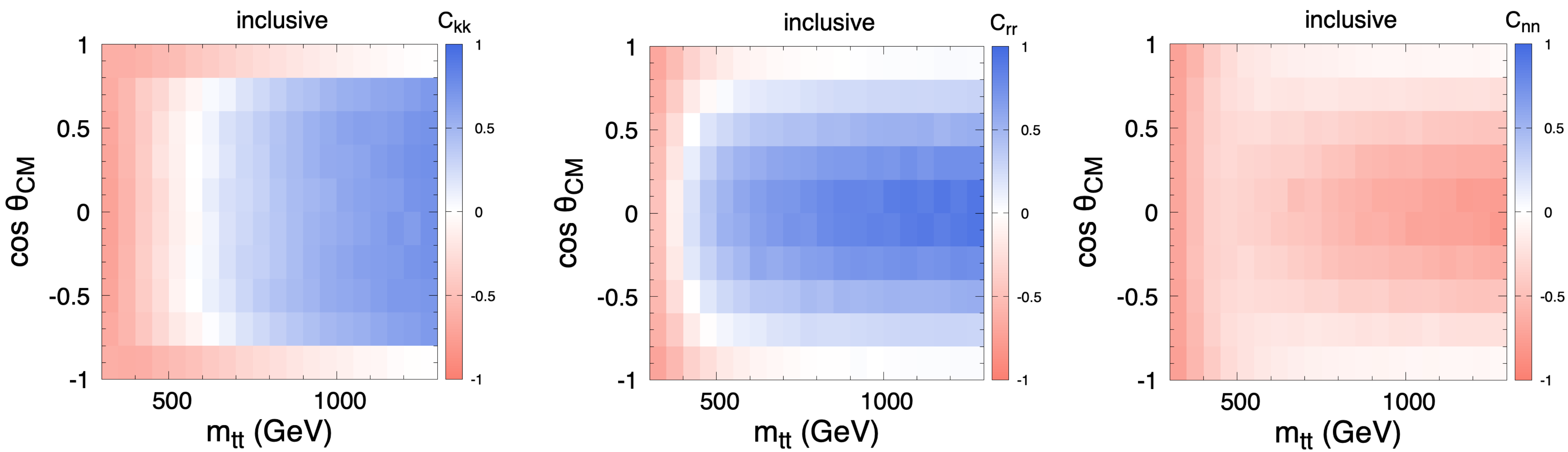
CHSH violation is probed by testing if $|C_{ii} \pm C_{jj}| > \sqrt{2}$

These estimators are optimal when off-diagonal C_{ij} vanish

CHSH inequalities in top pair production

CHSH inequalities in top pair production

It is really hard to violate CHSH inequalities!



Having $|C_{ii} \pm C_{jj}| > \sqrt{2}$ requires two C's of order 0.7, which can only be achieved quite close to threshold, or in the very boosted central region.

low statistics even at HL-LHC

- Threshold: $C_{kk} + C_{nn}$ → but beamline basis slightly better
- Boosted: $C_{rr} - C_{nn}$

CHSH inequalities in top pair production

CHSH violation involves only two coefficients. Near threshold, it pays off to make two of them larger even if the third one is smaller.

Beamline basis: simply $\hat{x} = (1,0,0) \quad \hat{y} = (0,1,0) \quad \hat{z} = (0,0,1)$

really tight cut!

$m_{t\bar{t}} \leq 353 \text{ GeV}$

	σ	C_{kk}	C_{rr}	C_{nn}	C_{kr}	C_{xx}	C_{yy}	C_{zz}
no β cut	303 fb	-0.677	-0.562	-0.712	0.067	-0.719	-0.719	-0.506
β ≤ 0.8	181 fb	-0.743	-0.640	-0.761	0.052	-0.767	-0.767	-0.602

estimator nearly optimal

CHSH violation indicator

$B \equiv |C_{xx} + C_{yy}| - \sqrt{2} = 0.024$ —(β ≤ 0.8)→ 0.120

CHSH inequalities in top pair production

In the boosted central region the helicity basis is way better

really tight cut!

$$m_{t\bar{t}} \geq 1 \text{ TeV}\,, \quad |\cos\theta_{\mathrm{CM}}| \leq 0.2$$

σ	C_{kk}	C_{rr}	C_{nn}	C_{kr}	C_{xx}	C_{yy}	C_{zz}
23.3 fb	0.659	0.874	-0.760	0.037	-0.043	-0.043	0.878

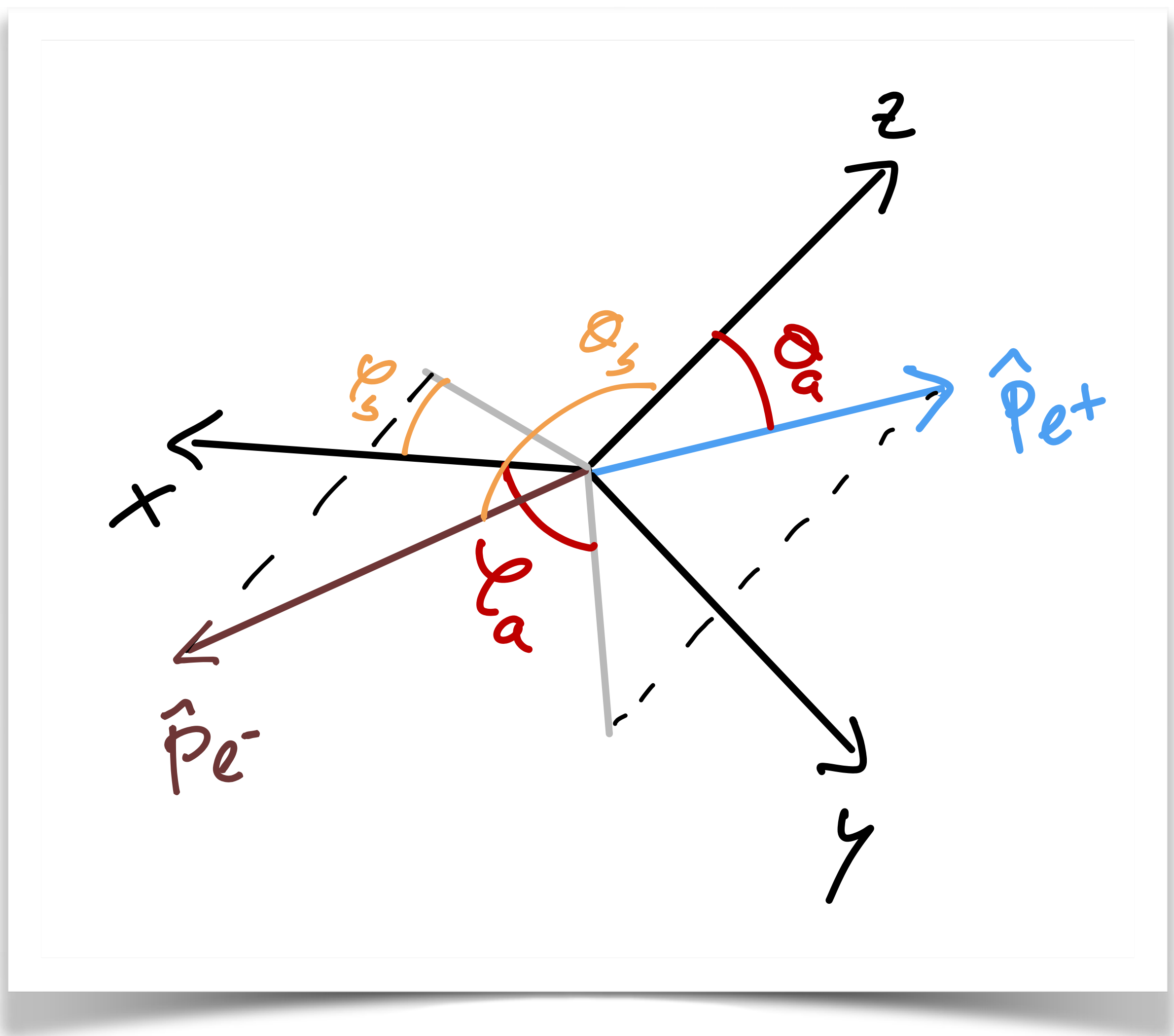
estimator nearly optimal

CHSH violation indicator

$$B \equiv |C_{rr} - C_{yy}| - \sqrt{2} = 0.210$$

What about dedicated observables to measure $|C_{ii} \pm C_{jj}|$?

JAAS, Casas,2205.00542



$$A_{\pm} = \frac{N(\cos(\varphi_a \mp \varphi_b) > 0) - N(\cos(\varphi_a \mp \varphi_b) < 0)}{N(\cos(\varphi_a \mp \varphi_b) > 0) + N(\cos(\varphi_a \mp \varphi_b) < 0)}$$

$$= \frac{\pi}{16}\alpha_a \alpha_b (C_{11} \pm C_{22})$$

How much is the improvement?

Setting systematics aside, there is an improvement of the statistical uncertainty of the `CHSH violation indicators´ $B = |C_{ii} \pm C_{jj}| - \sqrt{2}$

LHC Run 2+3 300 fb^{-1}

threshold [β, A_+]	$B:$	0.021 ± 0.053	$\longrightarrow$	0.121 ± 0.045	6.8 ×
boosted [A_-]	$B:$	0.218 ± 0.141	$\longrightarrow$	0.208 ± 0.125	1.13 ×

In all cases the statistics are small, therefore an improvement of the statistical sensitivity is very welcome.

HL-LHC 3 ab^{-1}

threshold [β, A_+]	$B:$	0.024 ± 0.017	$\longrightarrow$	0.124 ± 0.013	6.8 ×
boosted [A_-]	$B:$	0.218 ± 0.041	$\longrightarrow$	0.208 ± 0.036	1.13 ×

CHSH inequalities in top pair production

Loopholes? There are many!

- We are assuming quantum mechanics [in the decay of the top quark] to test quantum mechanics [by CHSH inequalities].
- Free-will loophole: we are not measuring spins in the pre-determined directions we want.
- Causal connection: near threshold, the decay of the two top quarks may not be causally disconnected.

… but in any case, these are very nice and demanding measurements!

Summary, again

- This topic [of quantum entanglement in HEP] does not seem to bring any new insight for new physics searches.
- But it is a nice twist in the interpretation of good old spin correlations in top - antitop, and more…
- And it motivates the introduction of new observables.
- It is quite fashionable now, and ATLAS and CMS are running to get the first measurements.
- Measurements will make nice headlines as the `highest-energy ever´ tests of quantum mechanics, etc.

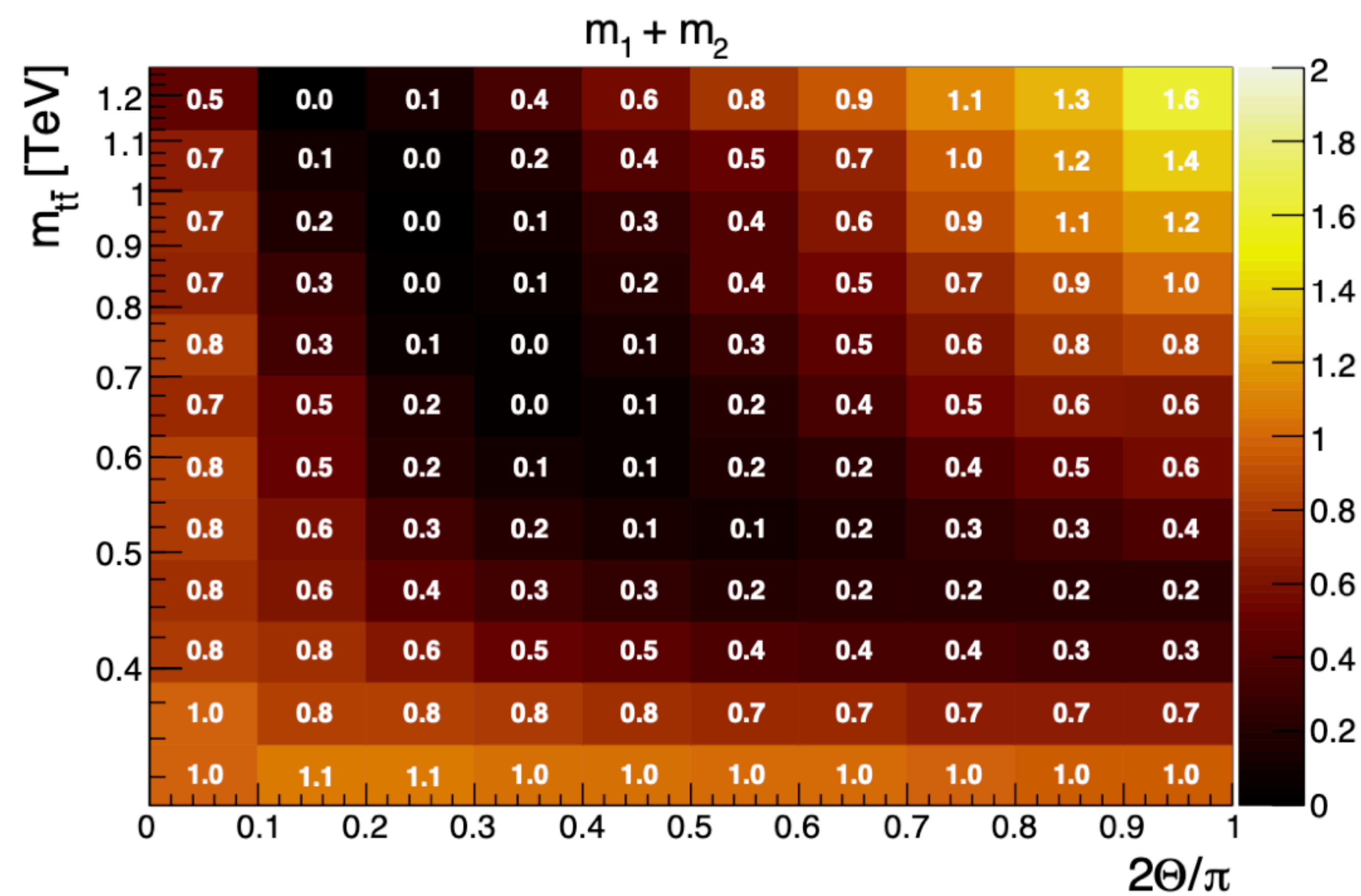
End

CHSH inequalities in top pair production

CHSH violation tests with C^TC eigenvalues [$\lambda_1 + \lambda_2 > 1$]

Fabbrichesi et al. 2102.11883

- Fast simulation with Delphes
- Only resonant diagrams
- Kinematical reconstruction and unfolding to parton level
- Optimised selection in m_{tt} – θ phase space



Helicity basis used BUT flipping axis for anti-top

CHSH violation at 98% CL [2.3σ] with Run 2 [139 fb^{-1}]

CHSH violation at 4σ with Run 3 [??? fb^{-1}]

Why do I worry about the flip sign in the basis definition? It is incorrect.

$$\rho = \begin{pmatrix} 1+C_{33} & 0 & 0 & C_{11}-C_{22}-i2C_{12} \\ 0 & 1-C_{33} & C_{11}+C_{22} & 0 \\ 0 & C_{11}+C_{22} & 1-C_{33} & 0 \\ C_{11}-C_{22}+i2C_{12} & 0 & 0 & 1+C_{33} \end{pmatrix}$$

In some phase space region:

Sign flip

$$C_{11} = 0.743 \quad C_{22} = 0.640$$
$$C_{33} = 0.761 \quad C_{12} = -0.052$$

→ Eigenvalues 1.9, 1.6, 1.6, -1.1

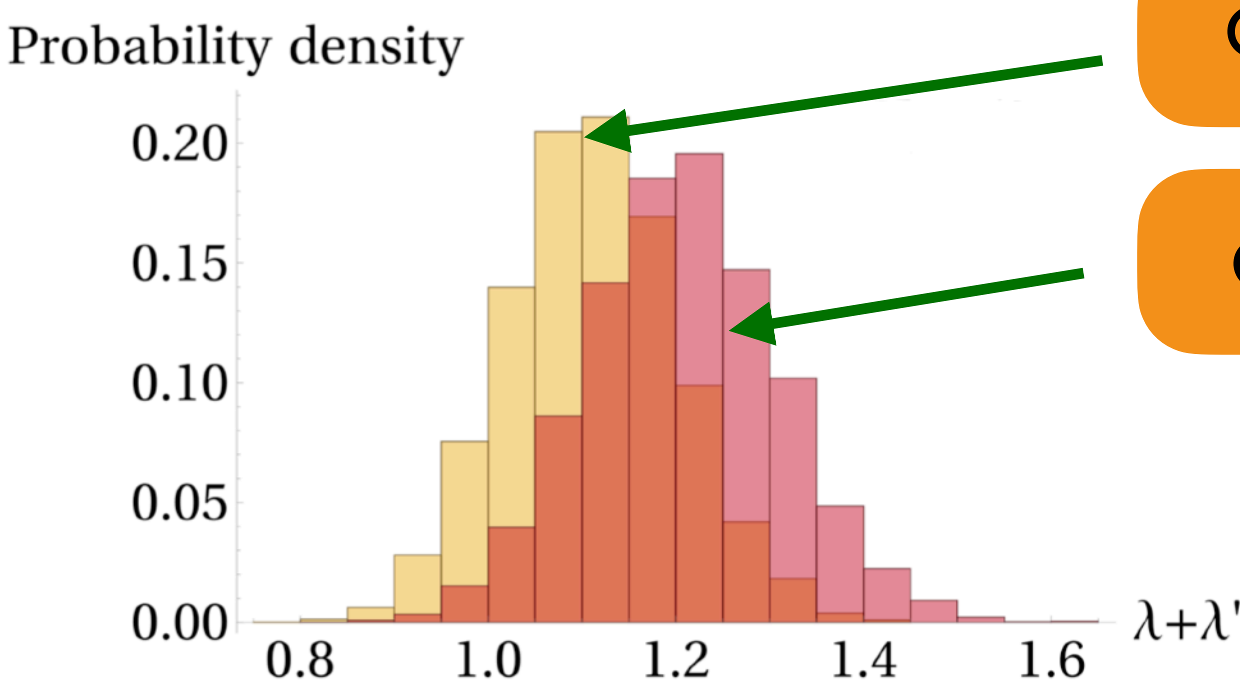
No sign flip

$$C_{11} = -0.743 \quad C_{22} = -0.640$$
$$C_{33} = -0.761 \quad C_{12} = 0.052$$

→ Eigenvalues 3.1, 0.4, 0.4, 0.1

CHSH inequalities in top pair production

Because the eigenvalues of $C^T C$ are biased, the violation of CHSH equalities is not signalled by $\lambda_1 + \lambda_2 > 1$ Severi et al. 2110.10112



CHSH-conserving has $\langle \lambda_1 + \lambda_2 \rangle > 1$

CHSH-violating

Significance decreases when taking bias into account

High-p_T Selection	$\lambda + \lambda'$ Parton–level	Significance for > 1 $[3\,\mathrm{ab}^{-1}]$
Weak	1.12	$1.9\,\sigma$
Intermediate	1.20	$2.1\,\sigma$
Strong	1.30	$1.3\,\sigma$