A new light on a family of hybrid algorithms

Miguel Murça

19 April 2023

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Simplifying a classical-quantum algorithm interpolation with quantum singular value transformations

Duarte Magano and Miguel Murça
Phys. Rev. A **106**, 062419 – Published 16 December 2022

https://arxiv.org/abs/2207.14810

Outline

- Crash course in Quantum Computing (as seen by someone doing algorithms who was previously in physics and for a physics audience)
- Quantum Phase Estimation
- "α-Eigenvalue Estimation"
- Conclusion

Crash course in Quantum Computing

(as seen by someone doing algorithms who was previously in physics and for a physics audience)

Crash course in Quantum Computing

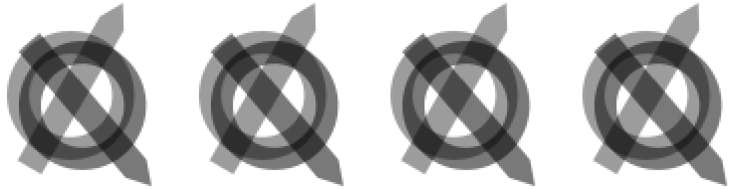
(as seen by someone doing algorithms who was previously in physics and for a physics audience)

→ What is a quantum computation?

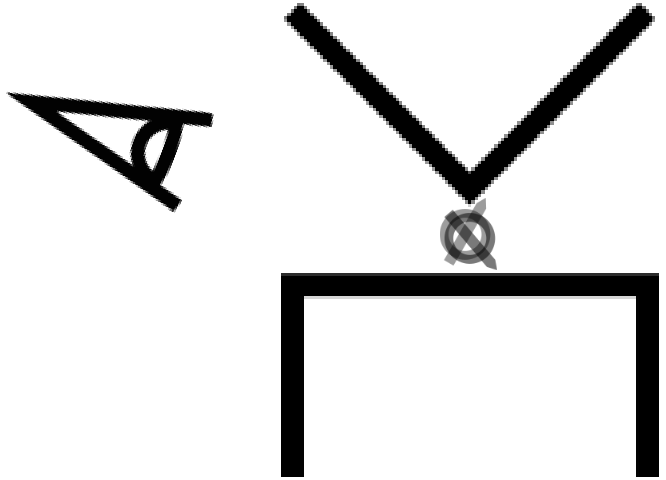
1.

2.

$$\xrightarrow{U}$$



3.



$$\begin{cases} \textsc{Up} & \longrightarrow 0 \\ \textsc{Down} & \longrightarrow 1 \end{cases}$$

Crash course in Quantum Computing

(as seen by someone doing algorithms who was previously in physics and for a physics audience)

→ What is a quantum computation?

$$U \quad \longleftrightarrow \quad \text{Yes/No question}$$

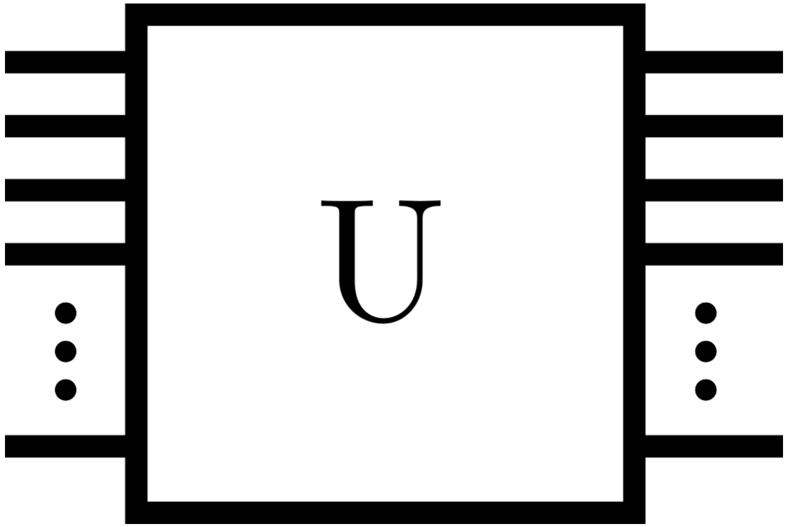
$$\begin{cases} \textsc{Up} & \longrightarrow 0 \\ \textsc{Down} & \longrightarrow 1 \end{cases} \quad \longleftrightarrow \quad \text{Yes/No answer}$$

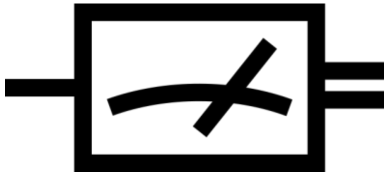
Crash course in Quantum Computing

(as seen by someone doing algorithms who was previously in physics and for a physics audience)

→ What is a quantum computation?

$$|0\rangle, |1\rangle$$





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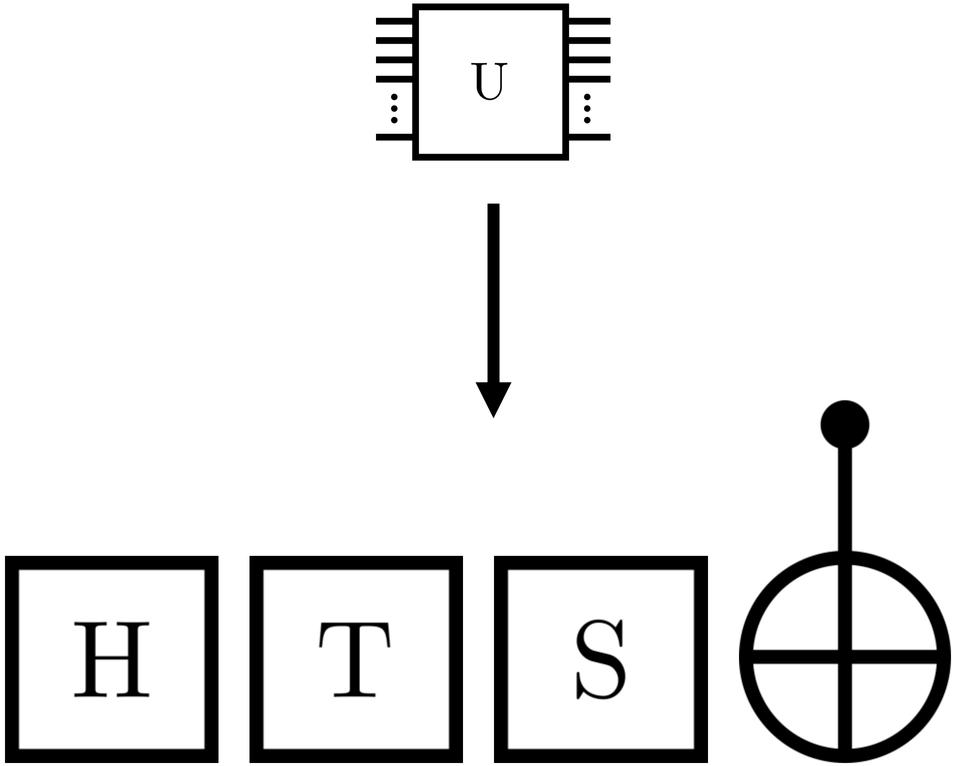

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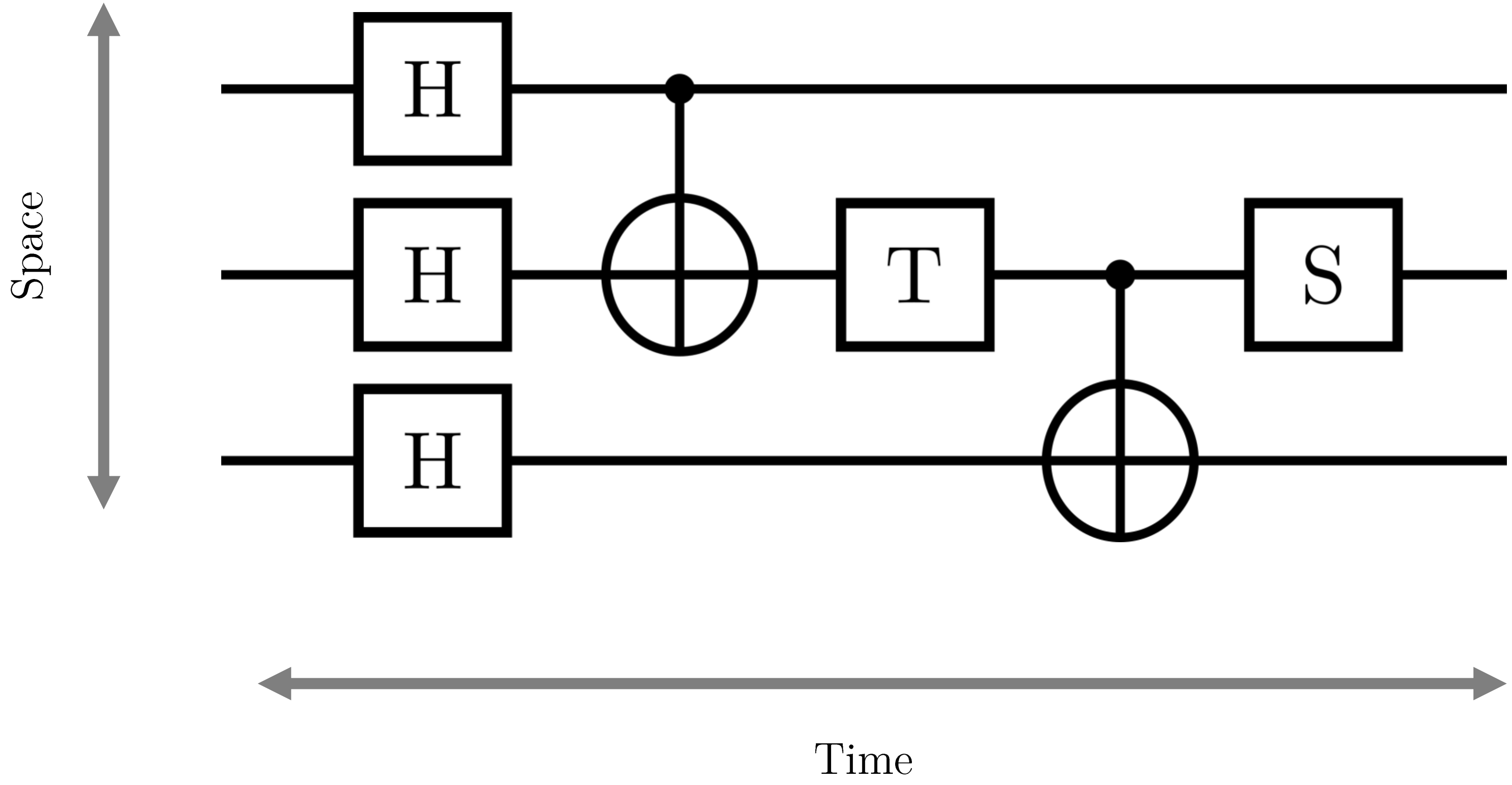

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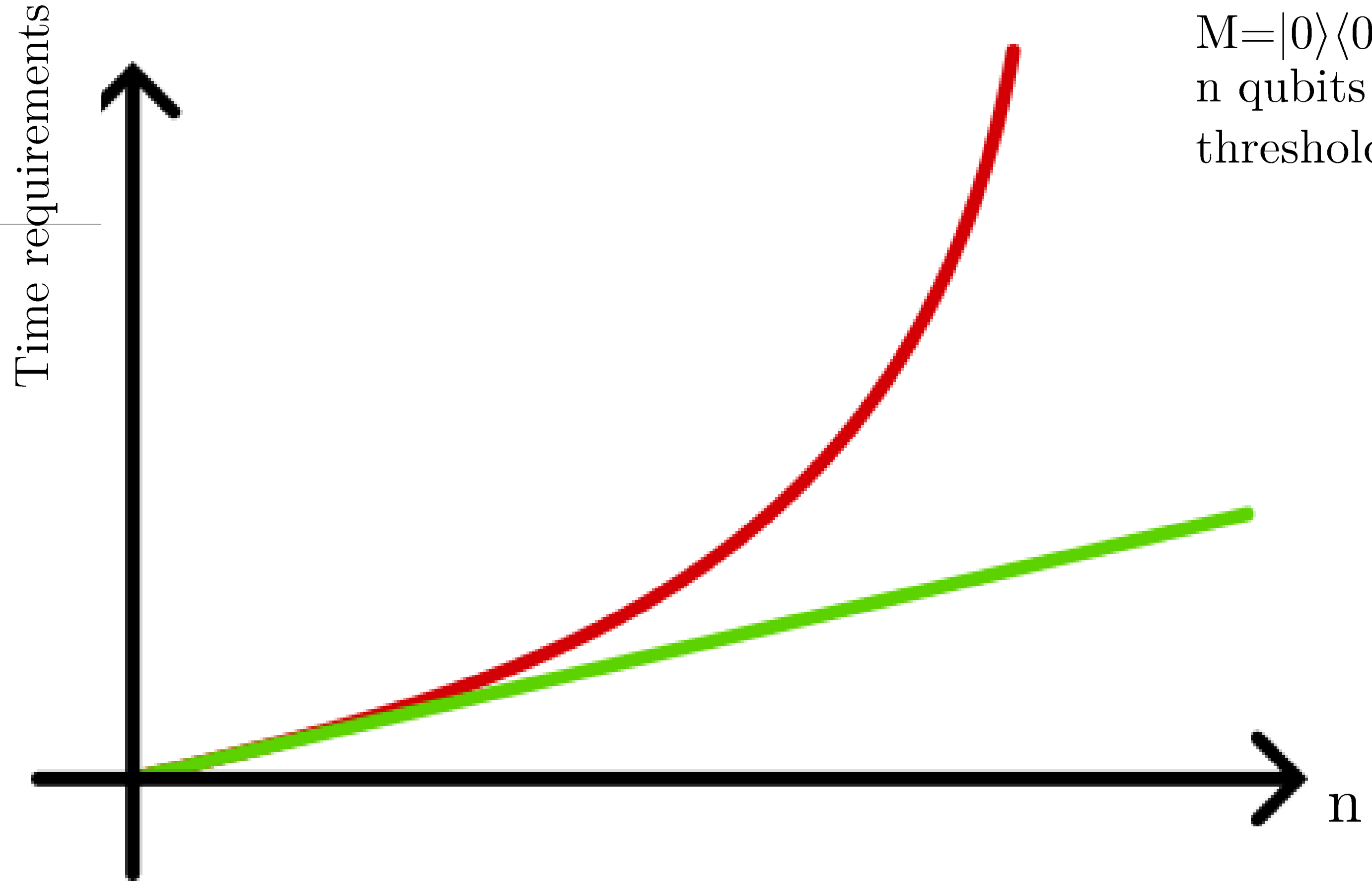
→ What is a quantum computation?

$U \longleftrightarrow$ Yes/No question

$\begin{cases} \text{Up} & \to 0 \\ \text{Down} & \to 1 \end{cases} \longleftrightarrow$ Yes/No answer

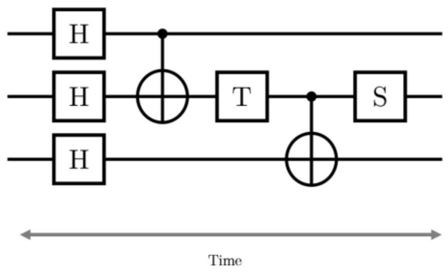


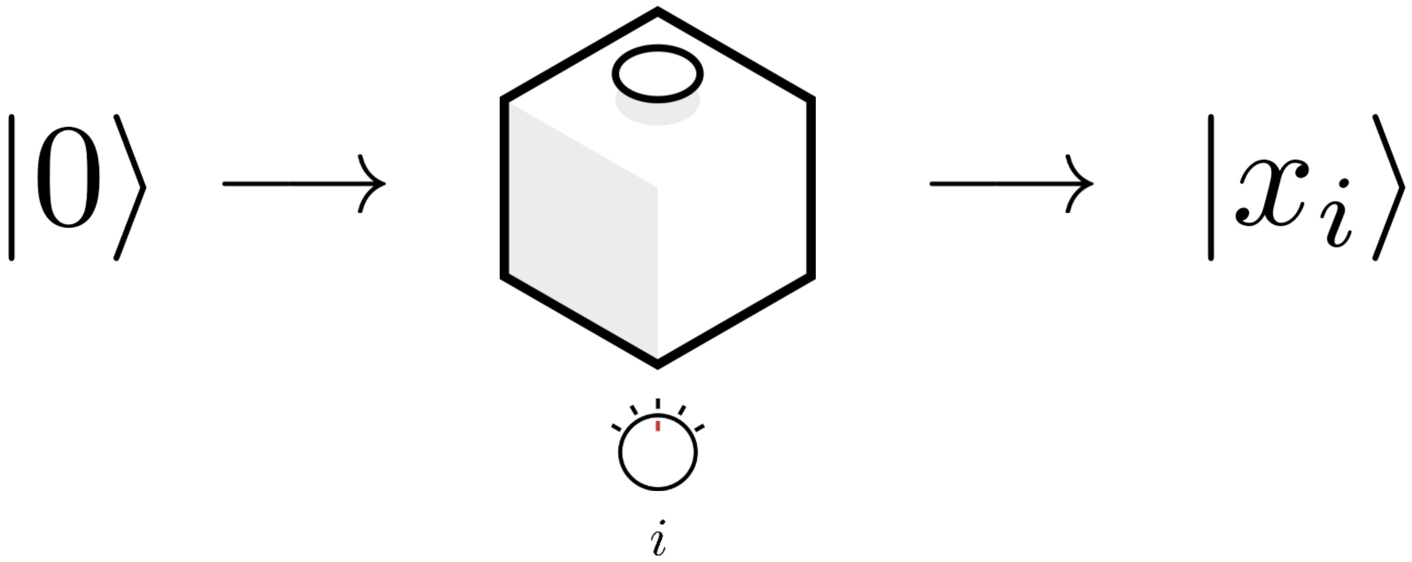
"Find whether the expectation of M=$|0\rangle\langle 0|$ under preparable $|\psi\rangle$ of n qubits is above or below a threshold for given n."

Crash course in Quantum Computing

(as seen by someone doing algorithms who was previously in physics and for a physics audience)

→ Oracle Model





$$O|0\rangle|i\rangle = |x_i\rangle|i\rangle$$

$U = U_T O U_{T-1} \cdots O U_1$ & $\langle 0 | U | 0 \rangle$ gives you the answer

Then T is the complexity of the algorithm.

- Good for proofs!
- May be a good description of your problem

Quantum Phase Estimation

Quantum Phase Estimation

Inputs:

U $2^n \times 2^n$ unitary operator

U_ψ to prepare $|\psi\rangle$, i.e., $U_\psi |0\rangle = |\psi\rangle$ (Formally: access to controlled U_ψ, $U_\psi^\dagger$)

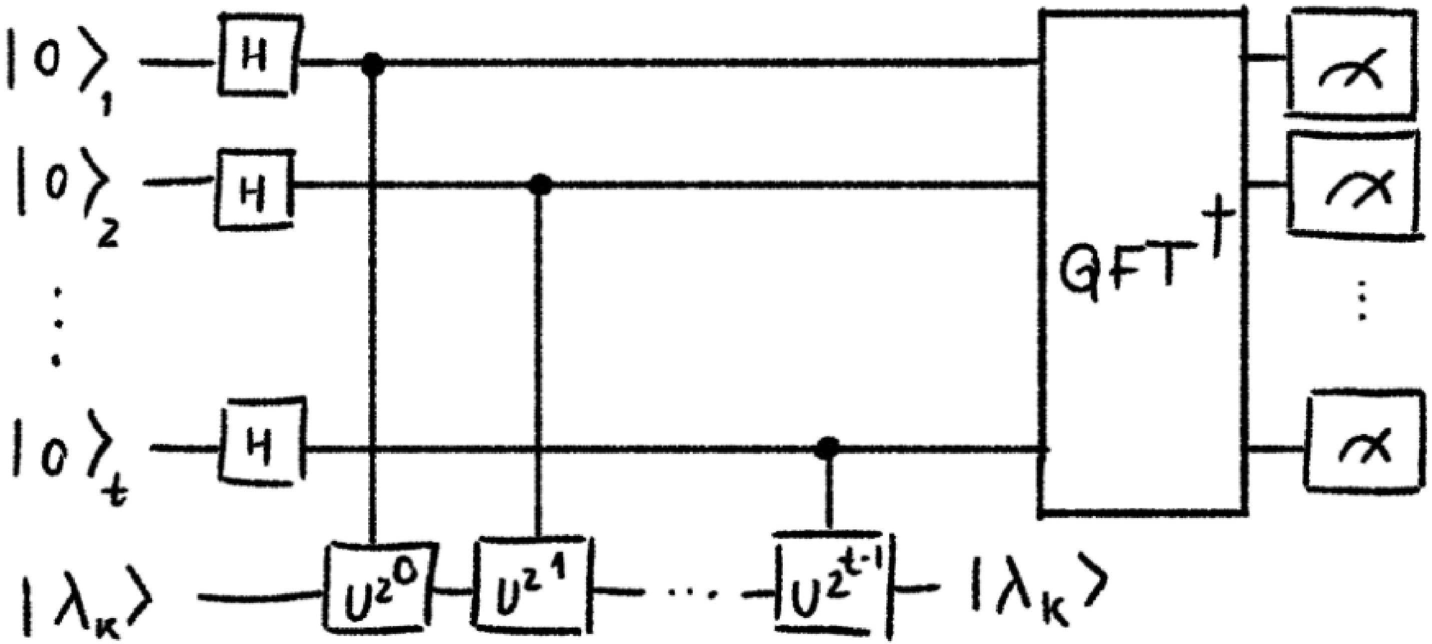
$|\psi\rangle$ $\in \mathbb{C}^{2^n}$ such that $U|\psi\rangle = e^{i\phi}|\psi\rangle$ $\phi \in [0, 2\pi)$

ϵ > 0 precision parameter

Outputs: ϕ up to precision ϵ with bounded error probability

Quantum Phase Estimation

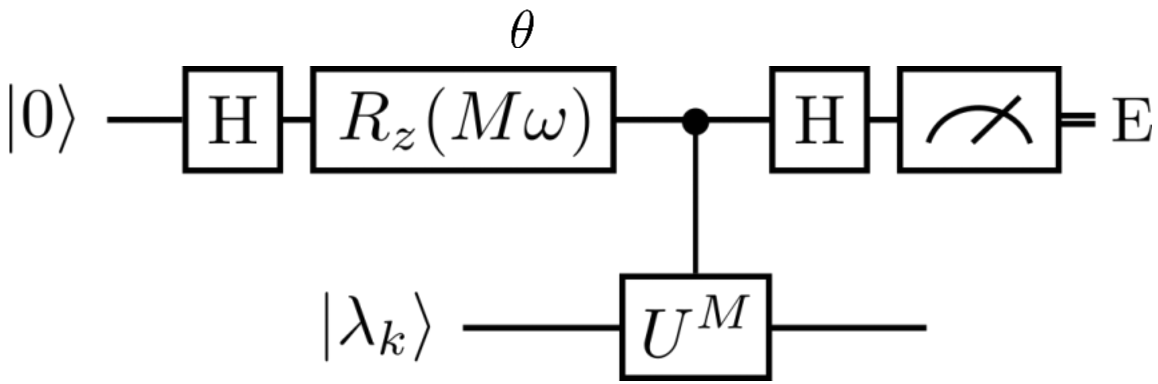
→ Textbook QPE Algorithm



(Quadratic oracle advantage)

Quantum Phase Estimation

→ Kitaev's/Iterative Phase Estimation Algorithm

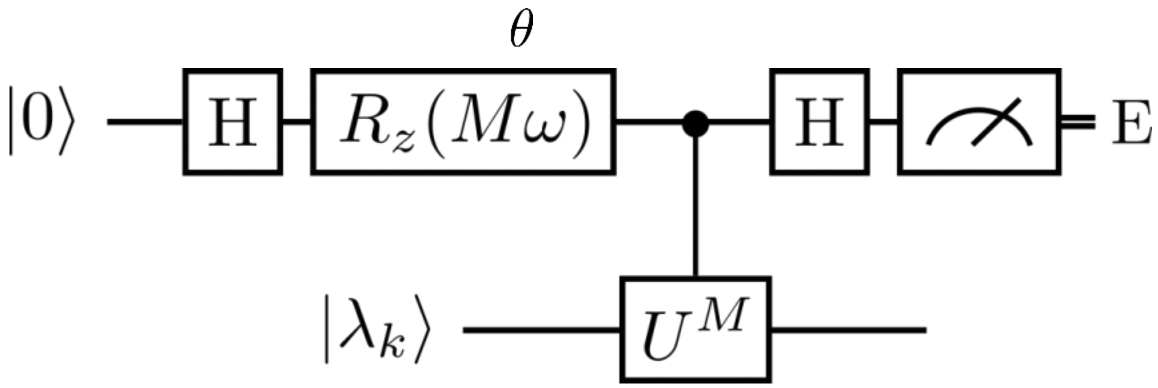


$$(M,\theta)_1 = (2^{m-1}, 0)$$
$$(M,\theta)_2 = (2^{m-2}, -\pi \cdot 2^{-m} \cdot \phi_m)$$
$$(M,\theta)_3 = (2^{m-3}, -\pi \cdot (2^{-m} \cdot \phi_m + 2^{-m+1} \cdot \phi_{m-1}))$$
$$\vdots$$
$$(M,\theta)_m = (1, -\pi \cdot (2^{-m} \cdot \phi_m + \cdots + 2^{-2} \cdot \phi_2))$$

$$\phi = 0.\phi_1\phi_2 \ldots \phi_m \ldots$$
$$\downarrow$$
$$\phi = 0.\phi_m \ldots$$
$$\downarrow$$
$$\phi = 0.\phi_{m-1}\phi_m \ldots - 0.0\phi_m$$
$$\downarrow$$
$$\ldots$$

Quantum Phase Estimation

→ Faster Phase Estimation



$$P(0|\phi;\theta,M) = \frac{1+\cos(M[\phi+\theta])}{2},$$
$$P(1|\phi;\theta,M) = \frac{1-\cos(M[\phi+\theta])}{2}.$$

[Submitted on 2 Apr 2013]

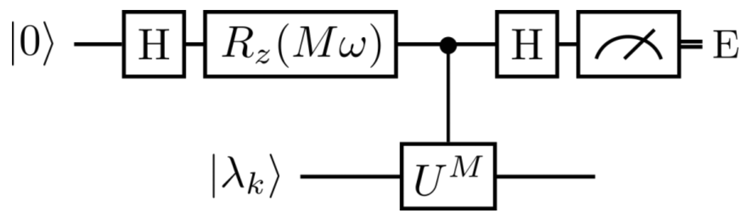
Faster Phase Estimation

Krysta M. Svore, Matthew B. Hastings, Michael Freedman

Idea: new "schedules" for M and ω

Plus: Informational perspective

Quantum Phase Estimation



→ α-Quantum Phase Estimation

Efficient Bayesian Phase Estimation

Nathan Wiebe and Chris Granade
Phys. Rev. Lett. 117, 010503 – Published 30 June 2016

$$M \sim 1/\epsilon, \theta = \mu$$

Accelerated Variational Quantum Eigensolver

Daochen Wang, Oscar Higgott, and Stephen Brierley
Phys. Rev. Lett. 122, 140504 – Published 12 April 2019

$$(M, \theta) = \left(\frac{1}{\epsilon^{\alpha}}, \mu - \epsilon \right)$$

Low depth algorithms for quantum amplitude estimation

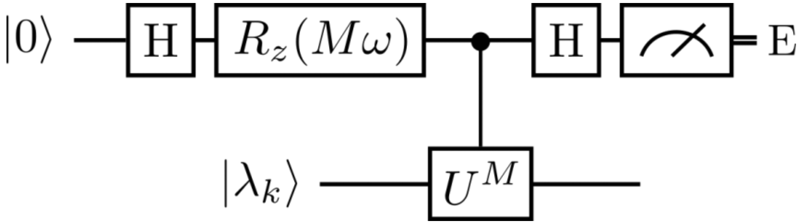
Tudor Giurgica-Tiron, Iordanis Kerenidis, Farrokh Labib, Anupam Prakash, and William Zeng
Quantum 6, 745 (2022)

$$N(\alpha) = \begin{cases} \frac{2}{1-\alpha}\left(\frac{1}{\epsilon^{2(1-\alpha)}} - 1\right) & \text{if } \alpha \in [0,1) \\ 4\log(1/\epsilon) & \text{if } \alpha = 1 \end{cases}$$

$$D(\alpha) = \mathcal{O}(1/\epsilon^{\alpha})$$

Quantum Phase Estimation

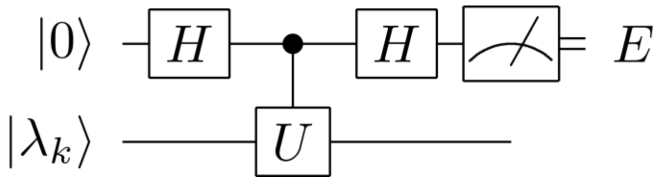
$\rightarrow$ Hybridization/Query Perspective



$$P(0|\phi;\theta,M) = \frac{1+\cos(M[\phi+\theta])}{2},$$
$$P(1|\phi;\theta,M) = \frac{1-\cos(M[\phi+\theta])}{2}.$$

$$P(x\,|\,\phi;\theta,M) = \frac{1+(-1)^x\cos(M[\phi+\theta])}{2}$$

Classical Sampling

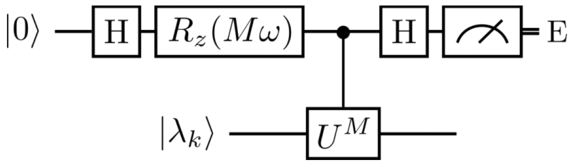


$D = \mathcal{O}(1)$ trivially

$N = \mathcal{O}(1/\epsilon^2)$ use Cramér-Rao bound

$T = N \times D = \sum D = \mathcal{O}(1/\epsilon^2)$

Fully Coherent



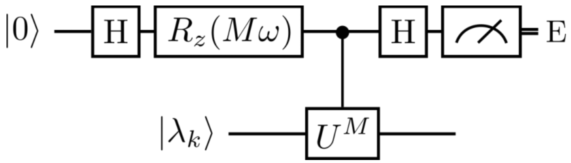
Plus Kitaev schedule

$D = \mathcal{O}(1/\epsilon)$

$T = \tilde{\mathcal{O}}(1/\epsilon)$

By inspection of the schedule

Hybrid (α-QPE)

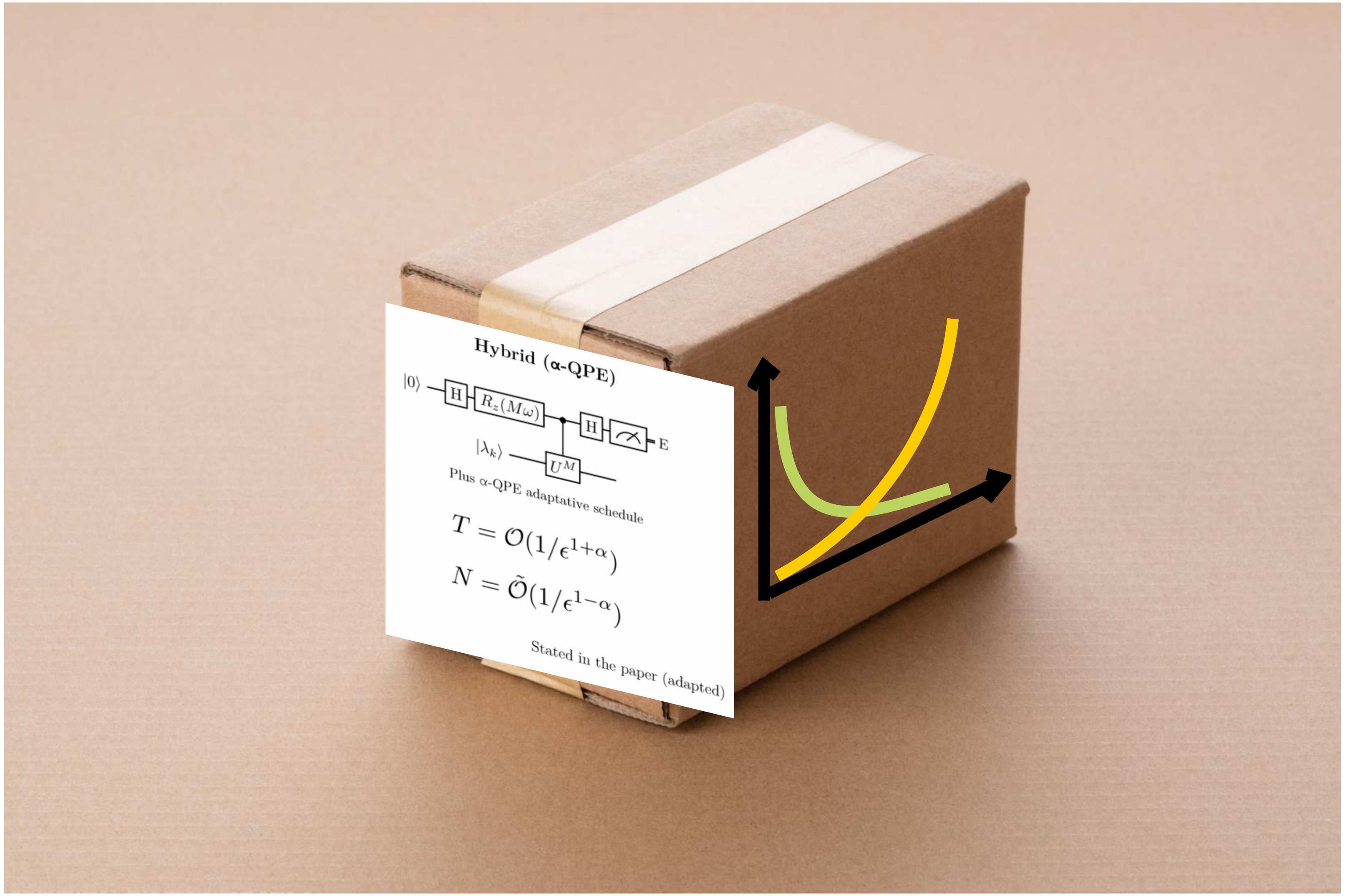


Plus α-QPE adaptative schedule

$D = \mathcal{O}(1/\epsilon^{1-\alpha})$

$T = \tilde{\mathcal{O}}(1/\epsilon^{1+\alpha})$

Stated in the paper (adapted)


Hybrid (α-QPE)
|0⟩
H
$R_z(M\omega)$
H
E
$|\lambda_k\rangle$
U^M
Plus α-QPE adaptative schedule
$T = \mathcal{O}(1/\epsilon^{1+\alpha})$
$N = \tilde{\mathcal{O}}(1/\epsilon^{1-\alpha})$
Stated in the paper (adapted)

α-Eigenvalue Estimation

α-Eigenvalue Estimation

→ Quantum Singular Value Transformations

1. Block-Encode a Matrix

$$U = \begin{bmatrix} A & * \\ * & * \end{bmatrix} = A \otimes |0\rangle\langle 0| + \cdots$$

2. Choose a suitable polynomial

$$p \in \mathbb{R}[x^d]$$
$$|p(x)| \leq 1 \text{ for } x \in [-1, 1]$$
$$p \text{ has partiy } d$$

3. Then with d applications of U you can produce

$$U' = \begin{bmatrix} p(A) & * \\ * & * \end{bmatrix} = \begin{bmatrix} \sum_\lambda p(\lambda) |\lambda\rangle\langle\lambda| & * \\ * & * \end{bmatrix}$$

$$A = \sum_\lambda \lambda |\lambda\rangle\langle\lambda|$$

α-Eigenvalue Estimation

→ "Phase Estimation is a weaker form of Eigenvalue Estimation"

Plan: $\text{PE} \preceq \text{AE} \preceq \text{EE}$

Define:

Amplitude Estimation

$$A|0^m\rangle = \sqrt{p}\,|\text{good}\rangle + \sqrt{1-p^2}\,|\text{bad}\rangle$$

$$O_A|\text{good/bad}\rangle = \pm|\text{good/bad}\rangle$$

Input: A, $A^\dagger$, O_A, $\epsilon > 0$

Output: $|p|$ up to ϵ with bounded error probability

Eigenvalue Estimation

$$\mathcal{H} \in \mathbb{H}_N \qquad \mathcal{H}|\psi\rangle = E|\psi\rangle$$

U_H is a (γ, m)–BE of $\mathcal{H}$

$$U_\psi|0^m\rangle = |\psi\rangle$$

Input: U_ψ, U_H, $U_H^\dagger$, γ, $\epsilon > 0$

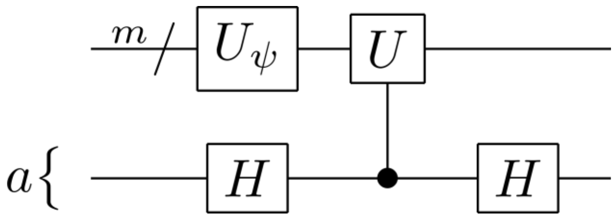
Output: E up to ϵ with bounded error probability

α-Eigenvalue Estimation

→ "Phase Estimation is a weaker form of Eigenvalue Estimation"

PE $\preceq$ AE

$A =$



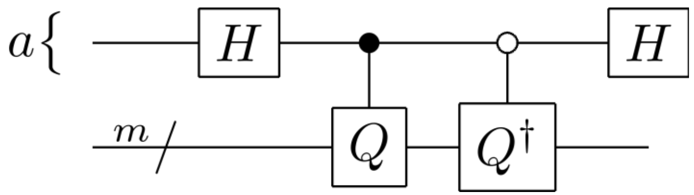
$$A\left|0^m\right\rangle\left|0\right\rangle = \cos(\phi/2)\left|\psi\right\rangle\left|0\right\rangle - i\sin(\phi/2)\left|\psi\right\rangle\left|1\right\rangle$$

↧

$$\cos(\phi/2) \rightsquigarrow |\phi|$$

AE $\preceq$ EE

$U_H =$



$$Q = A(2\left|0^m\right\rangle\!\left\langle 0^m\right| - I)A^\dagger O_A$$

$$\left\langle 0\right|_a U_H \left|0\right\rangle_a U_\psi \left|0^m\right\rangle = (1-2p)U_\psi\left|0^m\right\rangle$$

↧

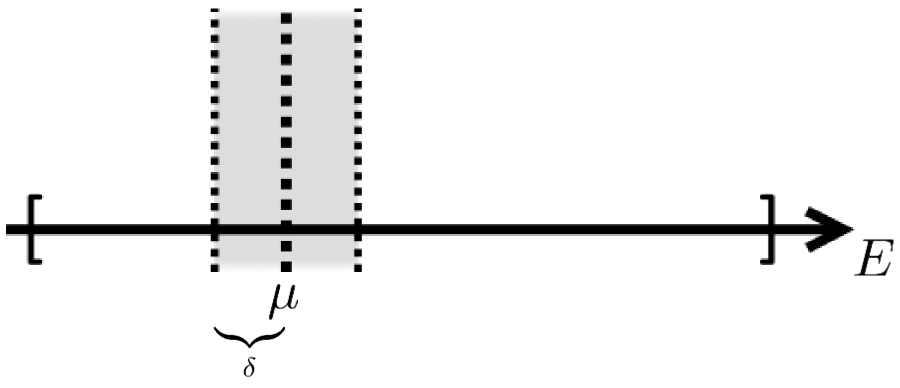
$$\mu = (1-2p) \rightsquigarrow p$$

α-Eigenvalue Estimation

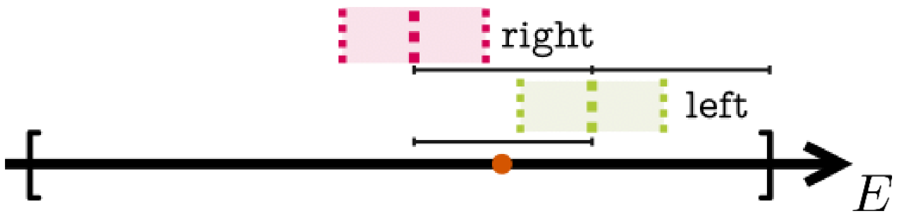
⇒ If we focus on EE (and find an interpolation for EE) we solve everything else

Now note:

If you can solve the decision problem



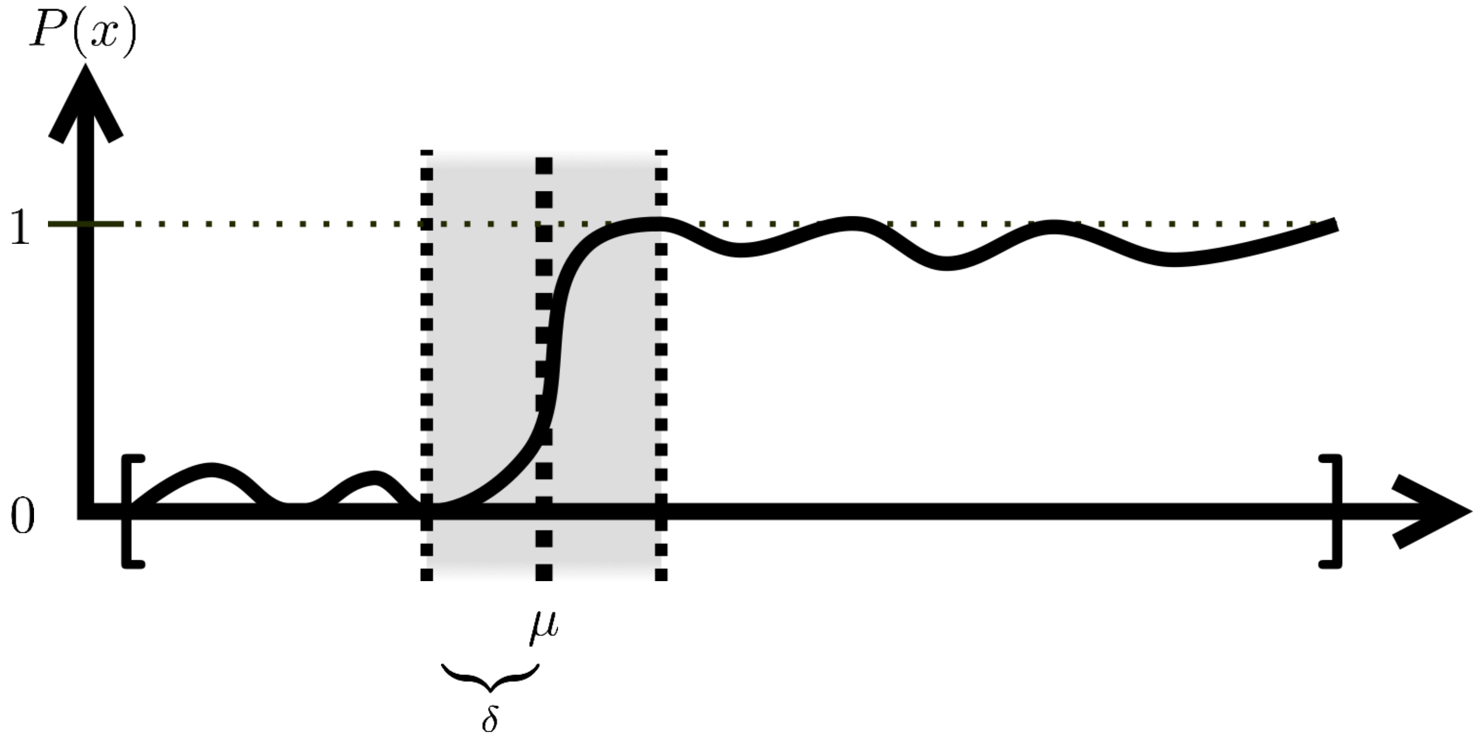
Then you can solve EE with a binary search



α-Eigenvalue Estimation

$$\begin{array}{c c} & \begin{array}{cc} |0\rangle & |1\rangle \end{array} \\ \begin{array}{c} \langle 0| \\ \langle 1| \end{array} & \begin{pmatrix} P(\mathcal{H}) & \cdot \\ \cdot & \cdot \end{pmatrix} \end{array} |0\rangle \otimes |\lambda_k\rangle \rightsquigarrow P(\lambda_k)\,|0\rangle\,|\lambda_k\rangle + |1\rangle \otimes (\cdots)$$

α-Eigenvalue Estimation



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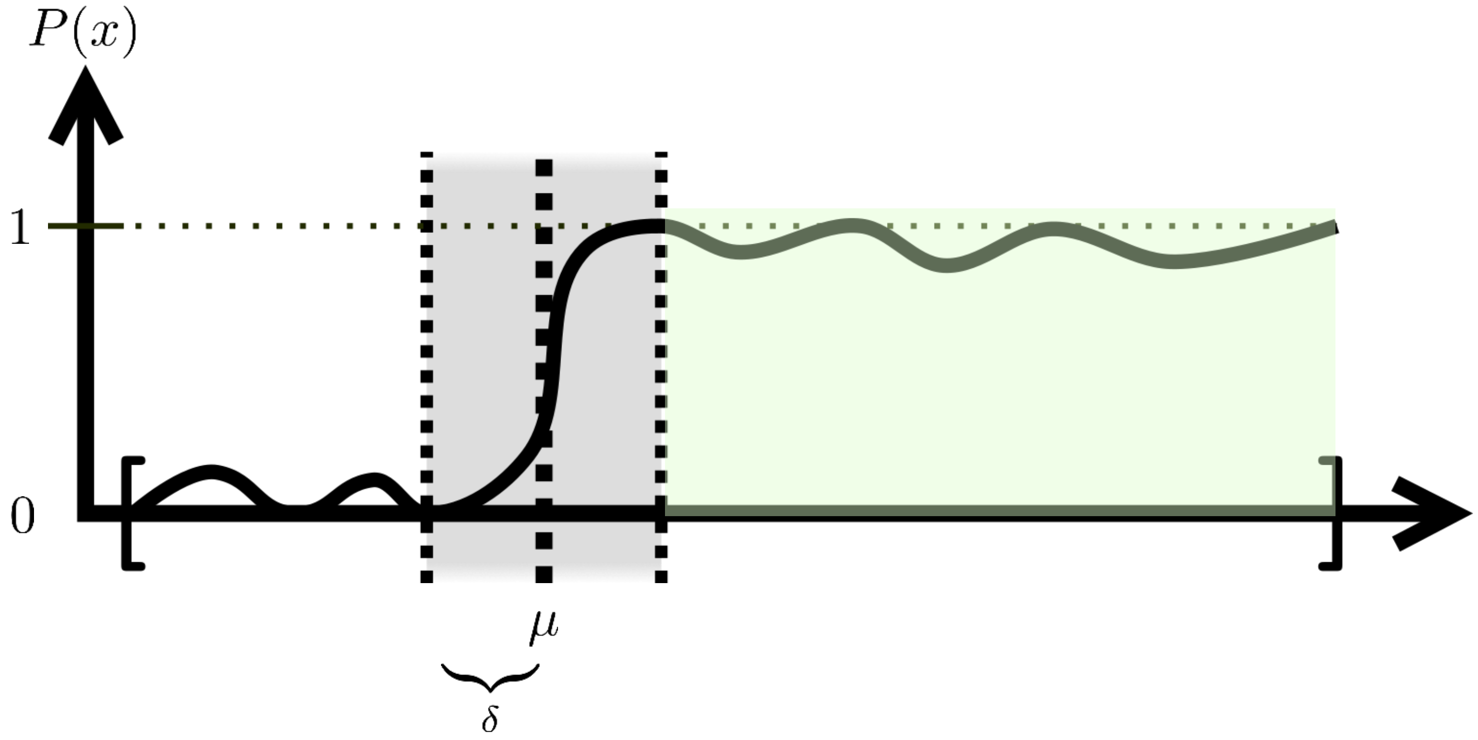

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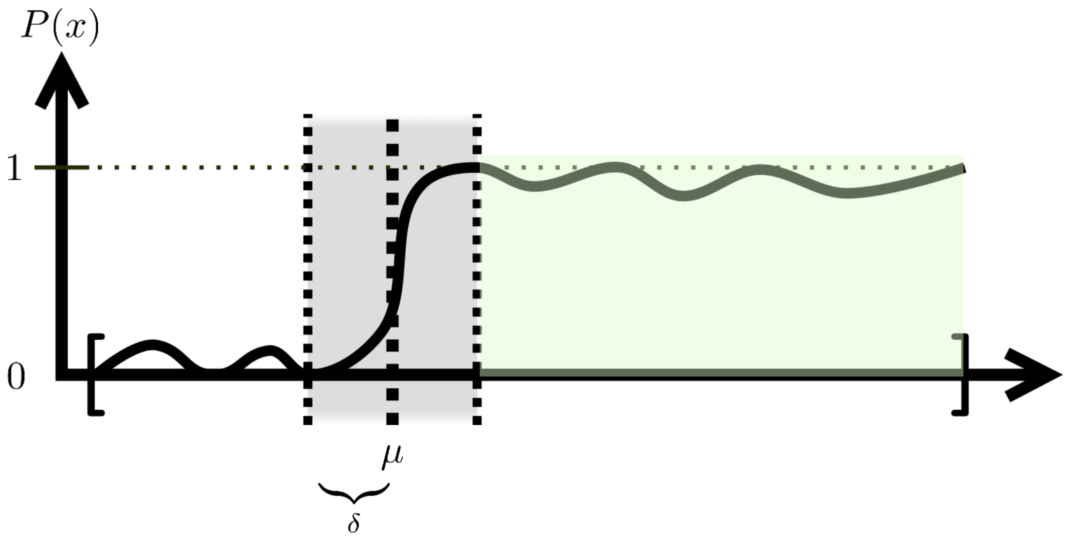
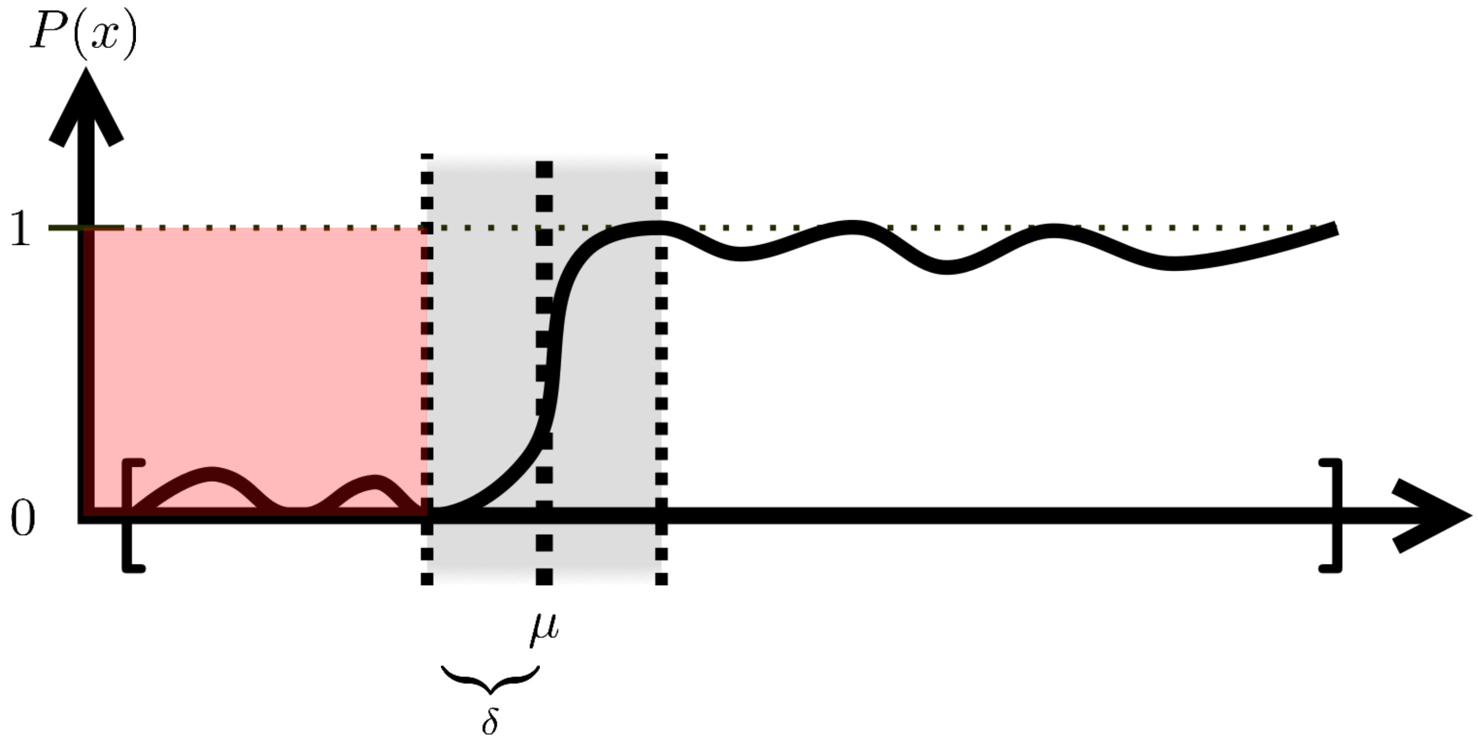


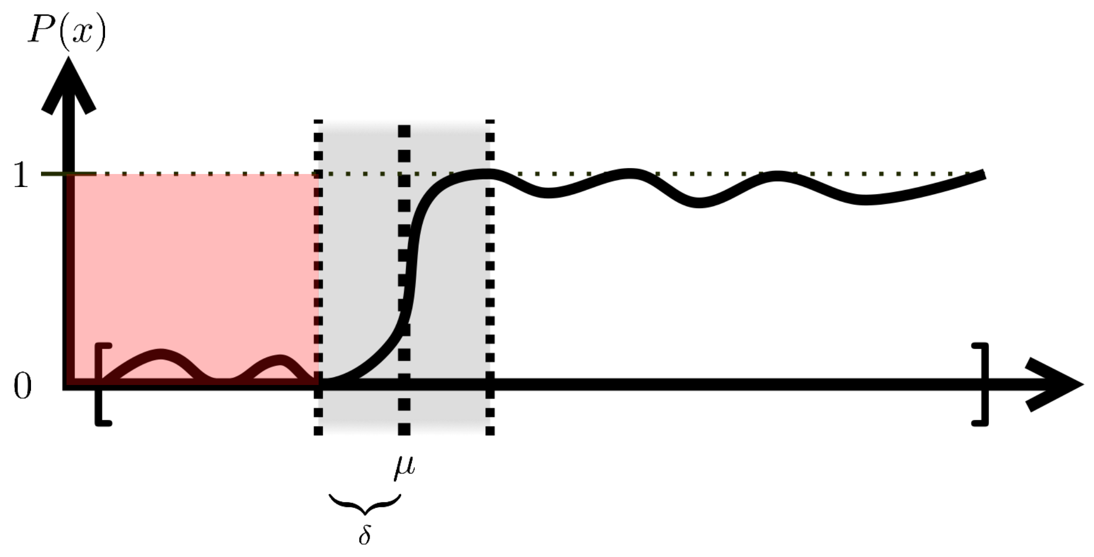
$$P(\lambda_k)\,|0\rangle\,|\lambda_k\rangle \approx |0\rangle\,|\lambda\rangle$$

α-Eigenvalue Estimation

→ QSVT



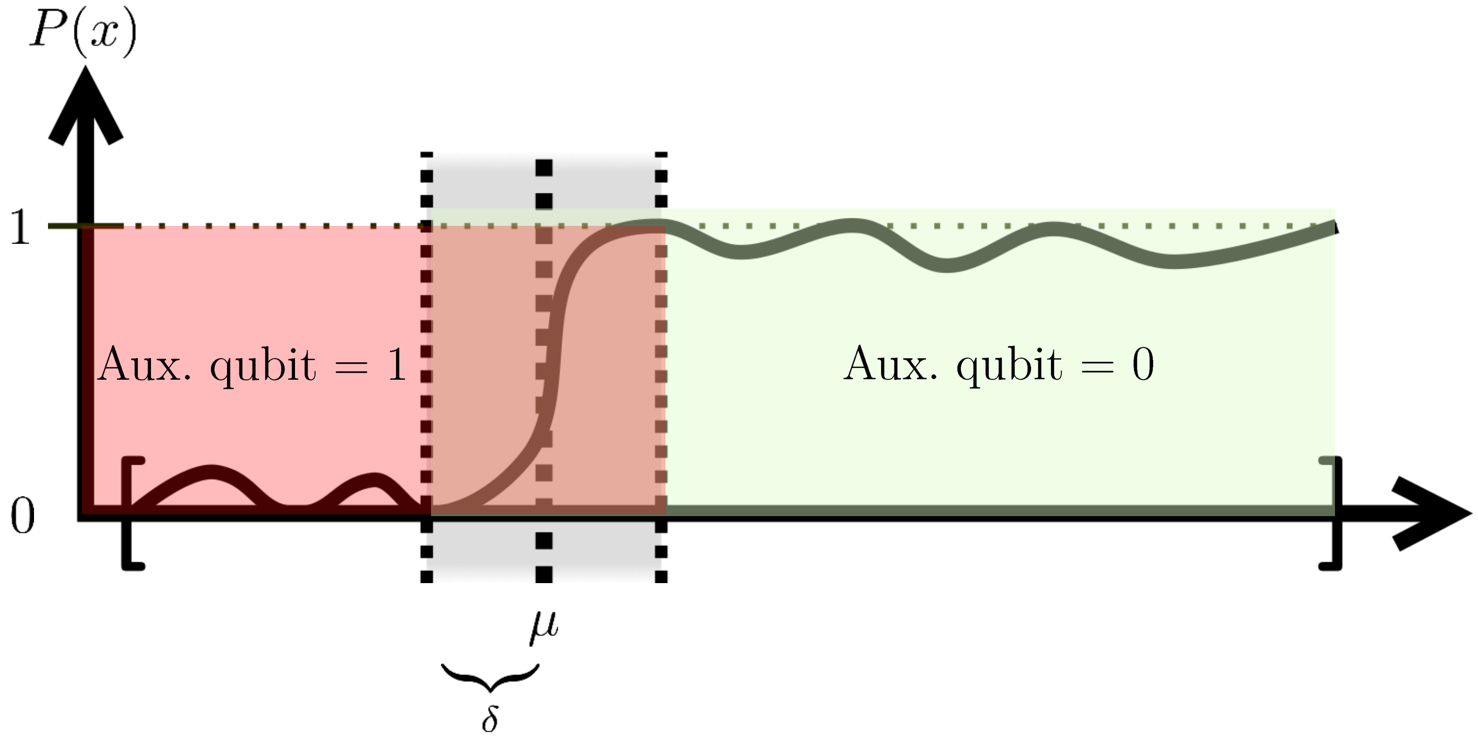
α-Eigenvalue Estimation



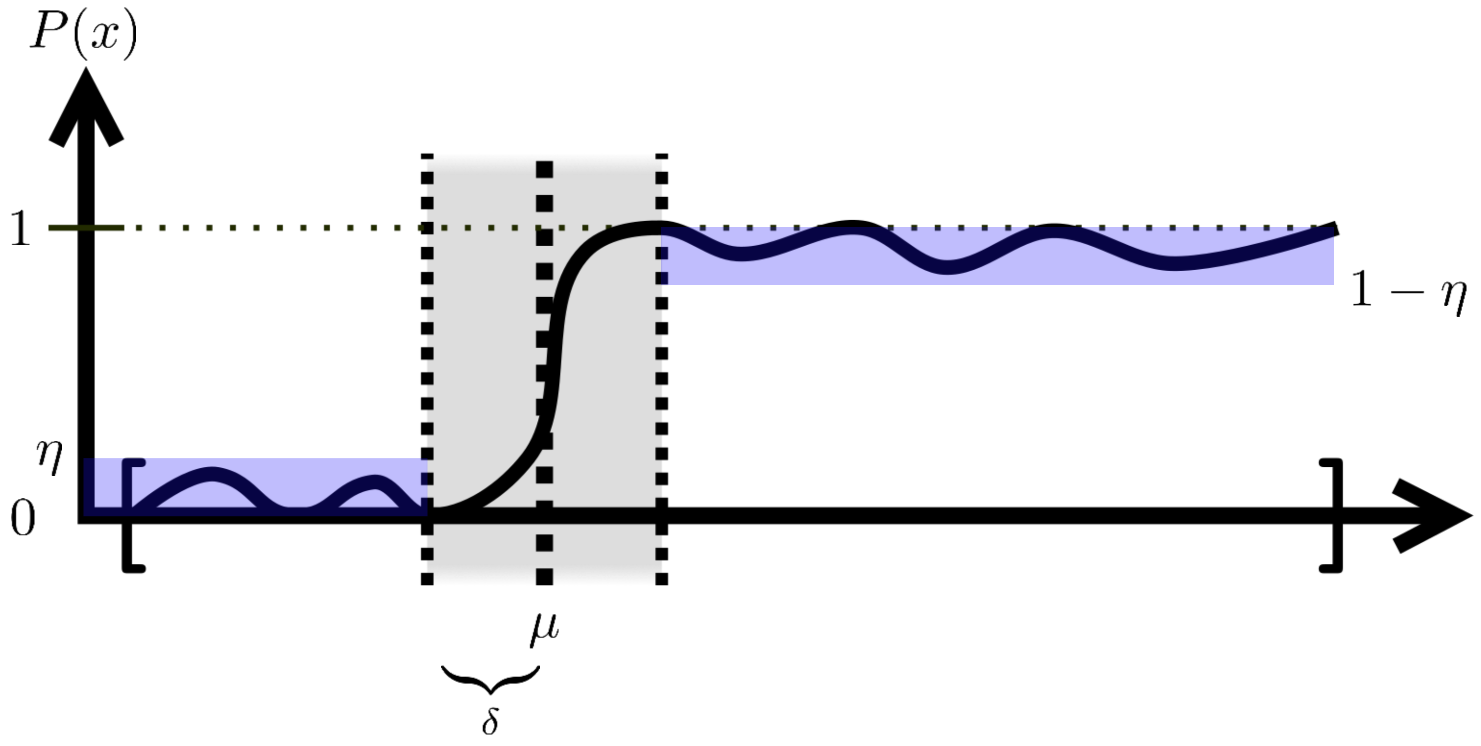
$$\begin{pmatrix} P(\mathcal{H}) & \cdot \\ \cdot & \cdot \end{pmatrix} = U_H |0\rangle |\lambda_k\rangle = P(\lambda_k) |0\rangle |\lambda_k\rangle + |1\rangle |u_j\rangle = |1\rangle |u_j\rangle$$

Must be unitary!

α-Eigenvalue Estimation



α-Eigenvalue Estimation



α-Eigenvalue Estimation

$\rightarrow$ Decision problem

$\Rightarrow$ If you can implement a step function, you can solve EE.

α-Eigenvalue Estimation

→ Decision problem

⇒ If you can implement a step function, you can solve EE.

… Can you implement a step function?

α-Eigenvalue Estimation

$\rightarrow$ Decision problem

$\Rightarrow$ If you can implement a step function, you can solve EE.

… Can you implement a step function?

Near-optimal ground state preparation

Lin Lin[1,2] and Yu Tong[1]

[1]Department of Mathematics, University of California, Berkeley, CA 94720, USA
[2]Computational Research Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA

Published: 2020-12-14, volume 4, page 372
Citation: Quantum 4, 372 (2020).

Yep! (Sort of)

α-Eigenvalue Estimation

→ Decision problem

Near-optimal ground state preparation

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Yep! (Sort of)

Published: 2020-12-14, volume 4, page 372
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Let U_H be a (γ, m)-block-encoding of a Hermitian matrix H and $\mu_0 \in [0, \gamma]$. Then, there is a $(1, m+3)$–block-encoding of $P\left(\frac{H-\mu_0 I}{\gamma+\mu_0}; \delta, \eta\right)$, where P satisfies

$$\forall x \in [-1, -\delta], 0 \leq P(x; \delta, \eta) \leq \eta/2 \quad (1)$$

$$\text{and } \forall x \in [\delta, 1], 1 - \eta/2 \leq P(x; \delta, \eta) \leq 1, \quad (2)$$

using $\mathcal{O}\left(\frac{1}{\delta} \log\left(\frac{1}{\eta}\right)\right)$ queries of U_H and $U_H^\dagger$.

(Adapted from lemma 5)

α-Eigenvalue Estimation

→ Decision problem

Let U_H be a (γ, m)-block-encoding of a Hermitian matrix H and $\mu_0 \in [0, \gamma]$. Then, there is a $(1, m+3)$–block-encoding of $P\left(\frac{H-\mu_0 I}{\gamma+\mu_0}; \delta, \eta\right)$, where P satisfies

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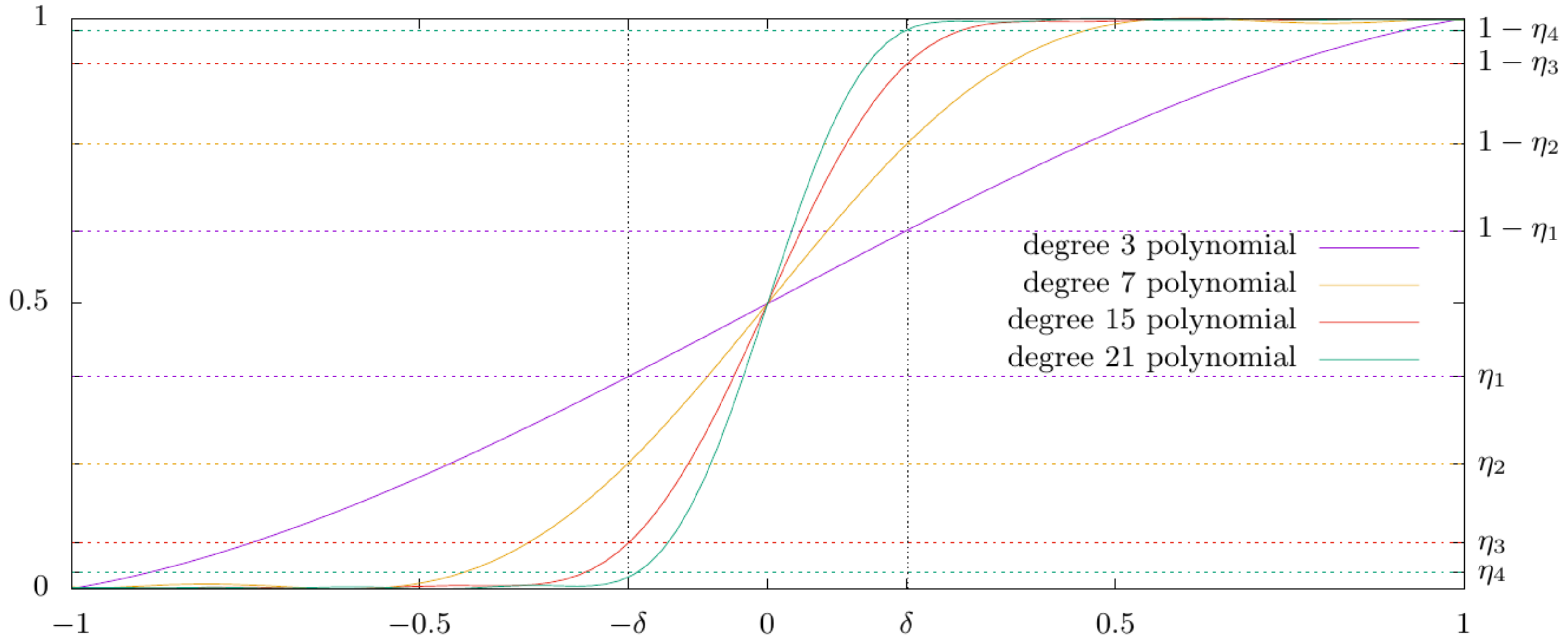
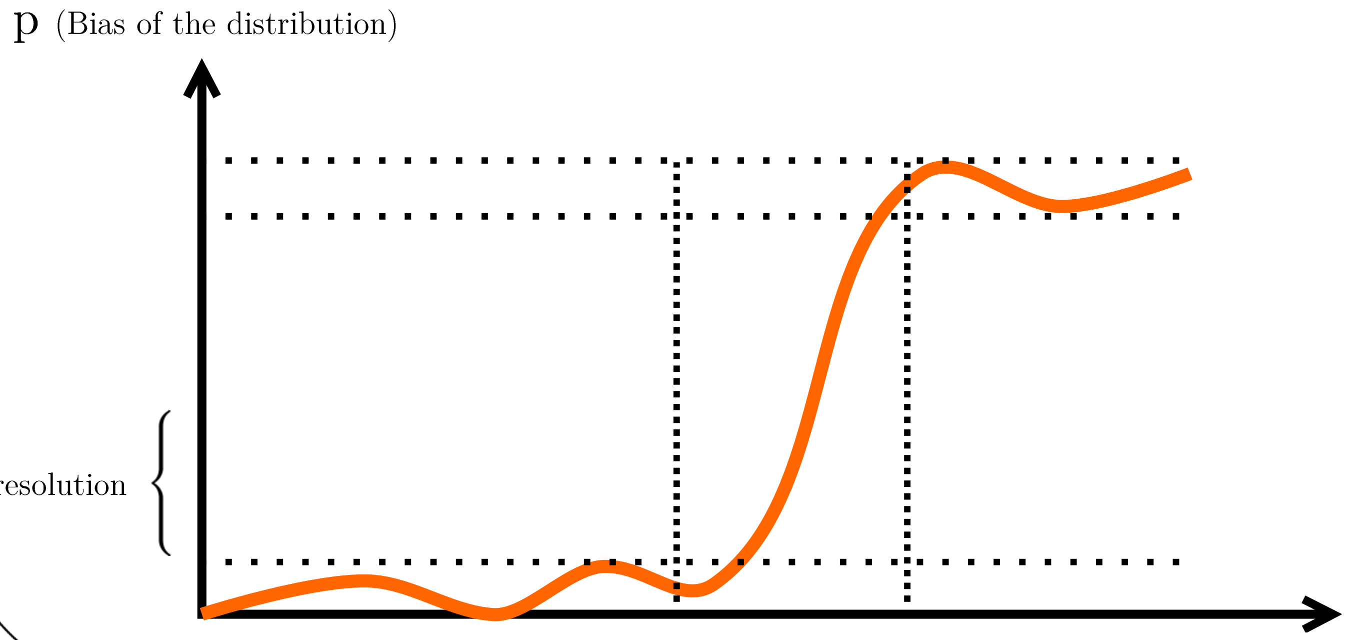


α-Eigenvalue Estimation

$\rightarrow$ Bringing it all together

What scalings do we get as a function of ε?



Need at least this resolution

$$\frac{1-2\eta}{2} \quad \underset{\Longrightarrow}{\text{by Chebyshev's inequality}} \quad N = \mathcal{O}\left(1/(1-2\eta)^2\right)$$

α-Eigenvalue Estimation

→ Bringing it all together

$$\eta = \frac{1}{2} - \frac{1}{4}\left(\frac{\epsilon}{4}\right)^{\alpha}$$

$$\delta = \epsilon/4$$

$$\implies \begin{cases} D(\alpha) = \mathcal{O}\left(\frac{1}{\epsilon}\log\left(\frac{1}{1-\epsilon^{\alpha}}\right)\right) = \mathcal{O}\left(\left(\frac{1}{\epsilon}\right)^{1-\alpha}\right) \\ N(\alpha) = \mathcal{O}\left(1/(1-2\eta)^2\right) = \mathcal{O}\left(D(\alpha)\left(\frac{1}{\epsilon}\right)^{2\alpha}\right) = \mathcal{O}\left(\left(\frac{1}{\epsilon}\right)^{1+\alpha}\right) \end{cases}$$

α-Eigenvalue Estimation

→ α-Eigenvalue Estimation

$$\implies \begin{cases} D(\alpha) = \mathcal{O}\left(\left(\frac{1}{\epsilon}\right)^{1-\alpha}\right) \\ T(\alpha) = \mathcal{O}\left(\left(\frac{1}{\epsilon}\right)^{1+\alpha} \log^2\left(\frac{1}{\epsilon}\right)\right) = \tilde{\mathcal{O}}\left(\left(\frac{1}{\epsilon}\right)^{1+\alpha}\right) \end{cases}$$

...which is the α-QPE scaling.

In Conclusion

- α-Quantum Phase Estimation can be "upgraded" to α-Eigenvalue Estimation
- You can think of the problem as "how well can I approximate a step function"
- Quantum Singular Value Transformations might be a good tool for finding hybrid algorithms
  - (Think: what do I need to do if I only have a poor approximation of my target function — and what *is* my target function)

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A new light on a family of hybrid algorithms

Miguel Murça

19 April 2023

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