The hidden gauge symmetry of relativistic dissipative hydrodynamics

or... Hydrodynamics with 50 particles. What does it mean and how to think about it?

G.Torrieri

2007.09224 (JHEP),
2109.06389 (With T.Dore,M.Shokri,L.Gavassino,D.Montenegro)
Answers somewhat speculative... but I think I am asking good questions!

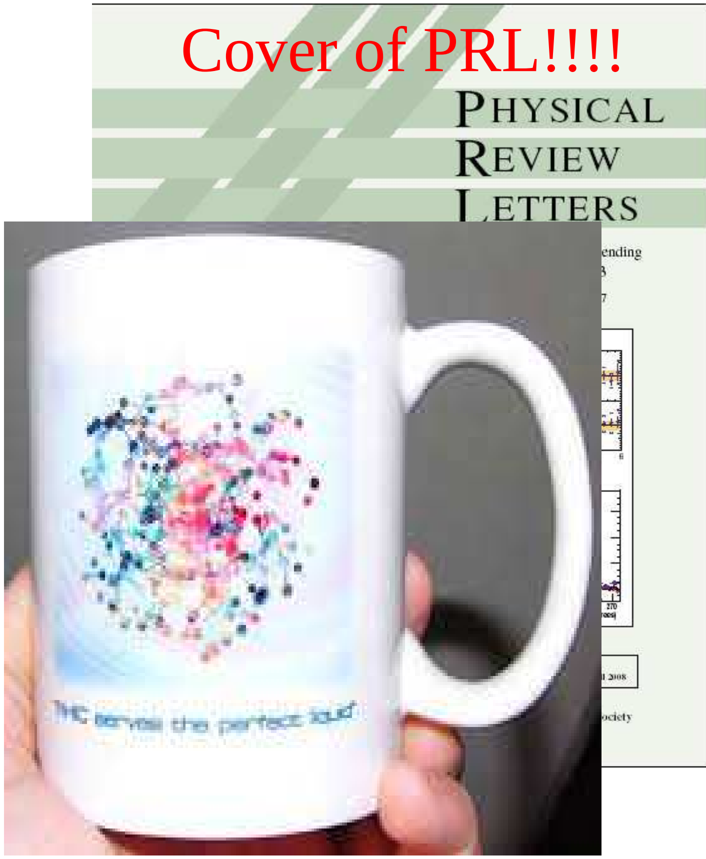
- The necessity to redefine hydro from stat.mechanics
  - Small fluids and fluctuations
  - Statistical mechanicists and mathematicians

- A possible answer:
  - Describing equilibrium at the operator level using the Zubarev operator
  - Definining non-equilibrium at the operator level using Crooks theorem

  Relationship to usual hydrodynamics analogous to "Wilson loops" vs "Chiral perturbation" regarding usual QCD

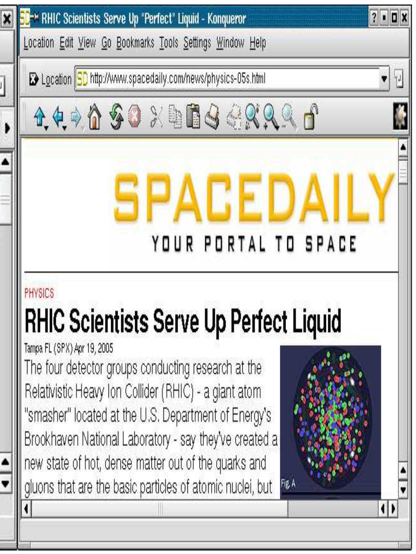
- The emergence of redundances and the reverse attractor . Fluctuations help thermalization, analogy with Gauge symmetry?

Cover of PRL!!!!

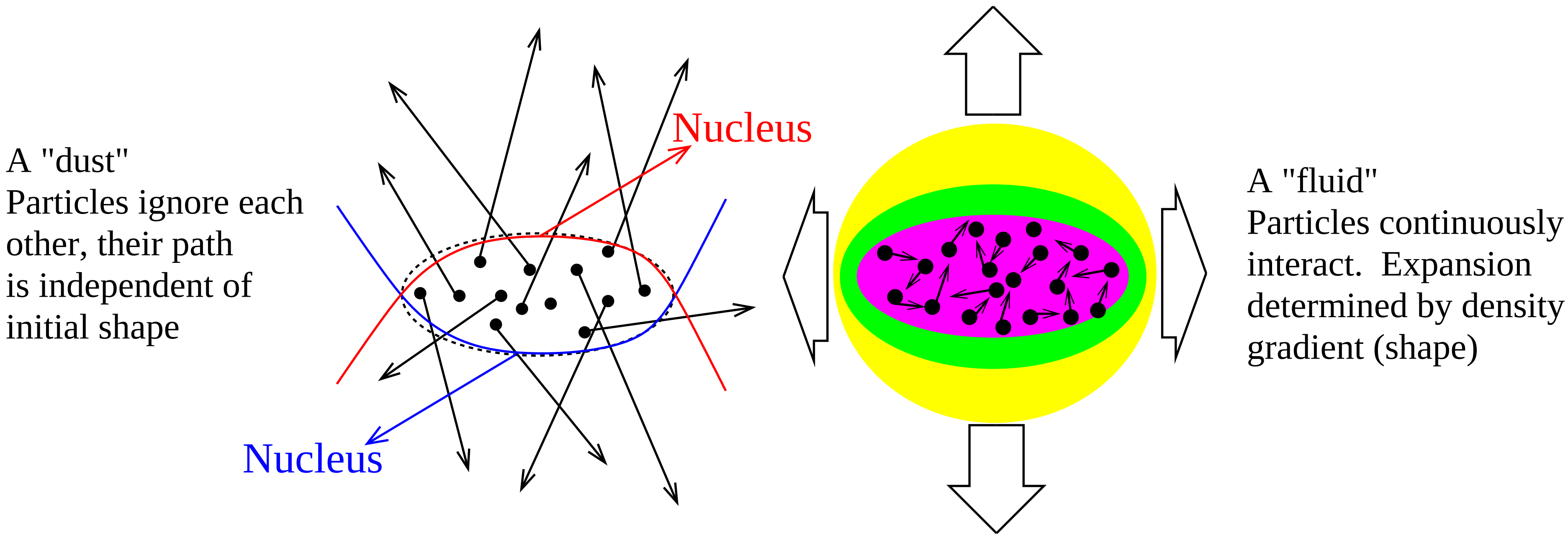


BBC!

SPACE DAILY!



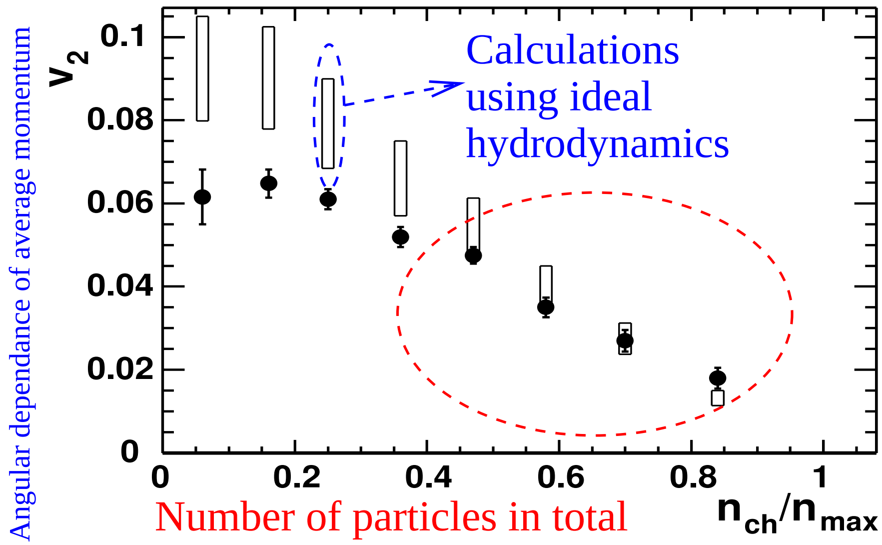
Heavy ion physicists found the perfect liquid! our field largely redefined to this



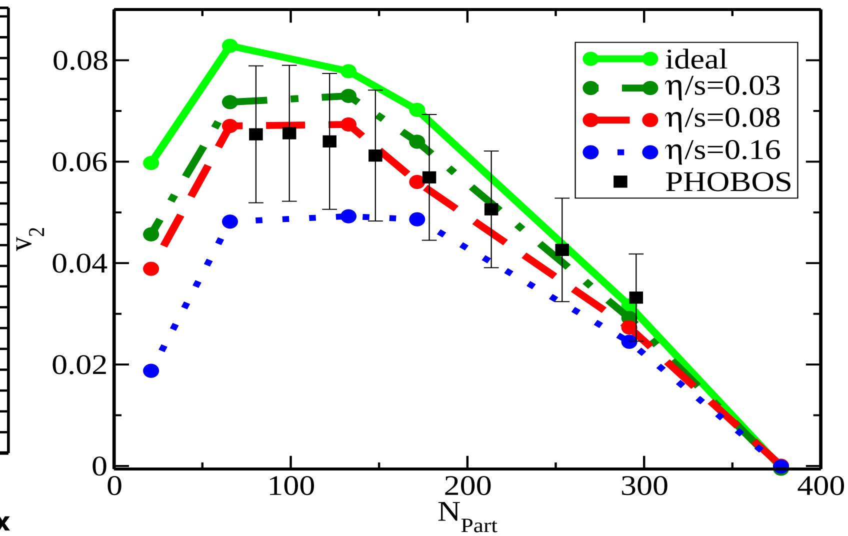
Observable: $\frac{dN}{p_T dp_T dy d\phi} = \frac{dN}{p_T dp_T dy}\left[1 + 2v_n(p_T, y)\cos\left(n\left(\phi - \phi_0\left(n, p_T, y\right)\right)\right)\right]$

"Collectivity" Same v_n appears in $\forall$ n-particle correlations , $\left\langle \frac{dN}{d\phi_1}\frac{dN}{d\phi_2}... \right\rangle$

P.Kolb and U.Heinz,Nucl.Phys.A702:269,2002.



P.Romatschke,PRL99:172301,2007



Fits ideal hydro , fitted upper limit on viscosity <u>low</u> Spurned <u>a lot</u> of theoretical and numerical/phenomenological development of relativistic hydrodynamics. Restarted the controversy over viscous relativistic hydrodynamics of the 70s

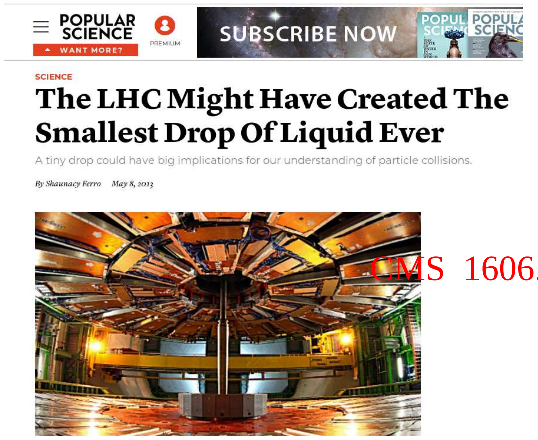
Conventional widsom: hydro EFT of gradients of conserved currents

$$\partial_\mu T^{\mu\nu}=0; T^{\mu\nu}= \underbrace{T^{\mu\nu}_{eq}}_{Thermal} + \underbrace{\Pi^{\mu\nu}}_{Relax} \equiv T^{\mu\nu}=T_0^{\mu\nu}(e,u)+\eta\mathcal{O}\left(\partial u\right)+\tau\mathcal{O}\left(\partial^2 u\right)+...$$

$$\eta=\lim_{k\to 0}\frac{1}{k}\mathrm{Im}\int dx\left\langle \hat{T}_{xy}(x)\hat{T}_{xy}(y)\right\rangle \exp\left[ik(x-y)\right] \quad , \quad \tau\sim\frac{\partial^2}{\partial k^2}\int e^{ikx}\left\langle TT\right\rangle ,$$

This is a classical theory , $\hat{T}_{\mu\nu}\to\langle T_{\mu\nu}\rangle$ Correlators $\langle T_{\mu\nu}(x)...T_{\mu\nu}\rangle$ play role in coefficients, not in EoM (if you know initial conditions, you know the whole evolution!) Kubo formula $w\to 0$ cuts out thermal fluctuations. Implicitly assumed$\ll$ mean free path

Both top-down ultimately derived from "microscopic" theories (Boltzmann equation,AdS/CFT), not "bottom up" statistical mechanics ("universality", independent from microscopic physics)!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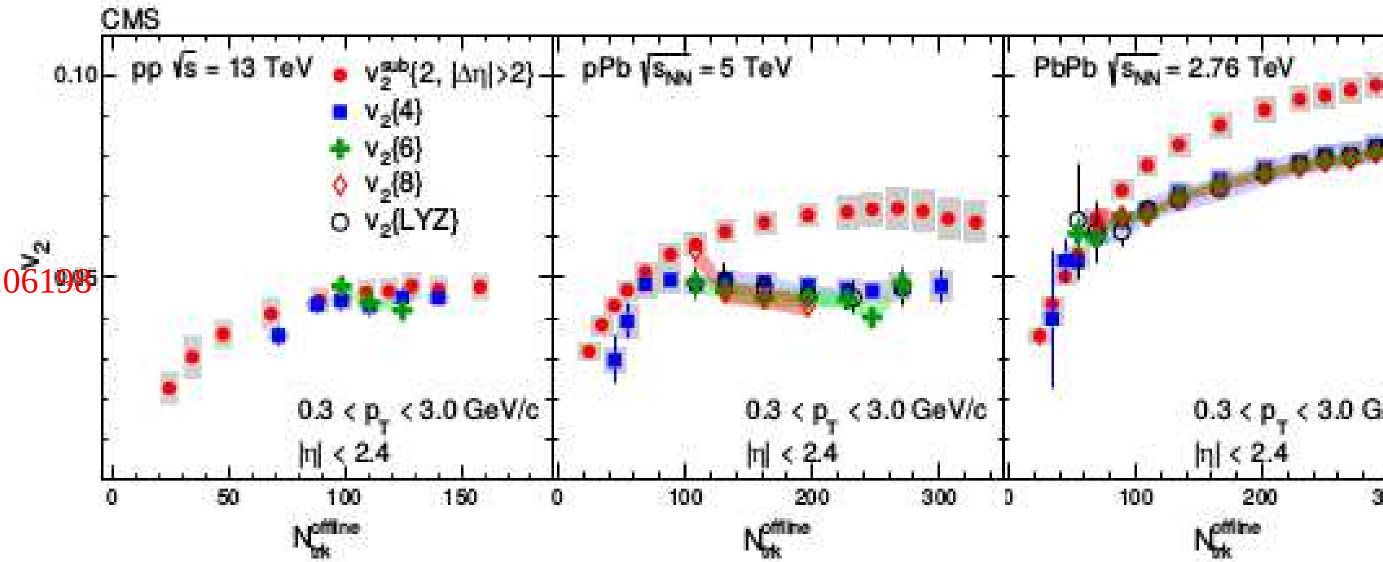
The LHC Might Have Created The Smallest Drop Of Liquid Ever

A tiny drop could have big implications for our understanding of particle collisions.

By Shaunacy Ferro May 8, 2013

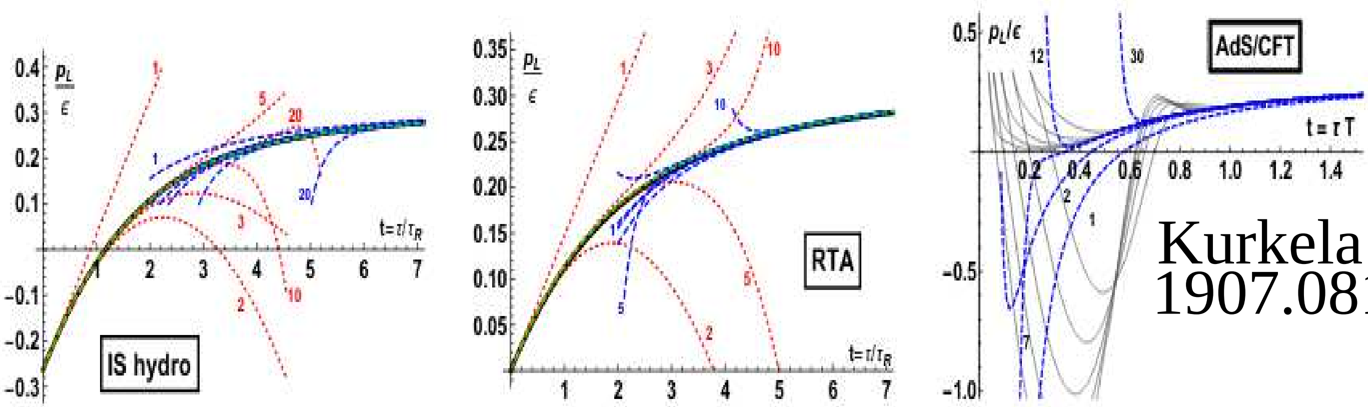
CMS 1606.06198



[1606.06198 (CMS)]: When you consider geometry differences and multi-particle cumulants (remove momentum conservation), hydro with $\mathcal{O}\left(20\right)$ particles <u>"just as collective"</u> as for 1000. Also cold atom fluids with 10,000 particles $\sim 1mm^3$

Little understanding of this in "conventional widsom". What si the smallest possible fluid?

Hydrodynamics in small systems: “hydrodynamization”/”fake equilibrium”
A lot more work in both AdS/CFT and transport theory about ”hydrodynamization”/”Hydrodynamic attractors”



Kurkela et al 1907.08101.

Fluid-like systems far from equilibrium (large gradients)! Usually from 1D solution of Boltzmann and AdS/CFT EoMs! “hydrodynamics converges even at large gradients with no thermal equilibrium”

But I have a basic question: ensemble averaging!

- What is hydrodynamics if $N \sim 50$ ...
  - Ensemble averaging , $\langle F(\{x_i\}, t)\rangle \neq F(\{\langle x_i\rangle\}, t)$ suspect for any non-linear theory. molecular chaos in Boltzmann, Large N_c in AdS/CFT, all assumed . But for $\mathcal{O}(50)$ particles?!?!
  - For water, a cube of length $\eta/(sT)$ has $\mathcal{O}\left(10^9\right)$ molecules,

$$P(N \neq \langle N\rangle) \sim \exp\left[-\langle N\rangle^{-1}(N - \langle N\rangle)^2\right] \ll 1$$

.

- How do microscopic, macroscopic and quantum corrections talk to eac other? EoS is given by $p = T \ln Z$ but $\partial^2 \ln Z/\partial T^2, dP/dV$??

NB: nothing to do with equilibration timescale . Even "things born in equilibrium" locally via Eigenstate thermalization have fluctuations!

And there is more... How does dissipation work in such a "semi-microscopic system"?

- What does local and global equilibrium mean there?
- If $T_{\mu\nu} \rightarrow \hat{T}_{\mu\nu}$ what is $\hat{\Pi}_{\mu\nu}$ Second law fluctuations? Sometimes because of a fluctuation entropy decreases! What is the role of microstates?

The obvious conclusion is Fluctuations only help dissipation, they are random .
Perhaps $l_{mfp} \geq \mathcal{O}(1) \left(V/N_{dof}\right)^{1/3}$ or something like this.

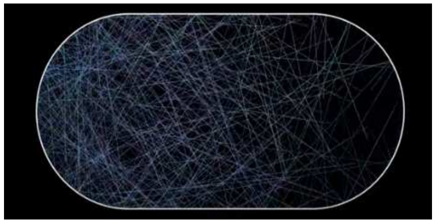
Can this be wrong? Can fluctuations help thermalize so smaller systems thermalize faster? if $1/T \sim l_{mfp}$? PERHAPS...

Bottom line: Either hydrodynamics is not the right explanation for these observables (possible! But small/big systems similar!) or we are not understanding something basic about what's behind the hydrodynamics! What do fluctuations do? Just a lower limit to dissipation?

Statistical mechanics: This is a system in global equilibrium, described by a partition function $Z(T,V,\mu)$, whose derivatives give expectation values $\langle E \rangle$,fluctuations $\langle (\Delta E)^2 \rangle$ etc. in terms of conserved charges. All microstates equally likely, which leads to preferred macrostates!

Fluid dynamics: This is the state of a field in local equilibrium which can be perturbed in an infinity of ways. The perturbations will then interact and dissipate according to the Euler/N-S equations. many issues connecting to Stat.Mech. Wild weak solutions, millenium problem!

The problem with general "transport thinking"

 =  ?

Let's solve the simplest transport equation possible: Free particles

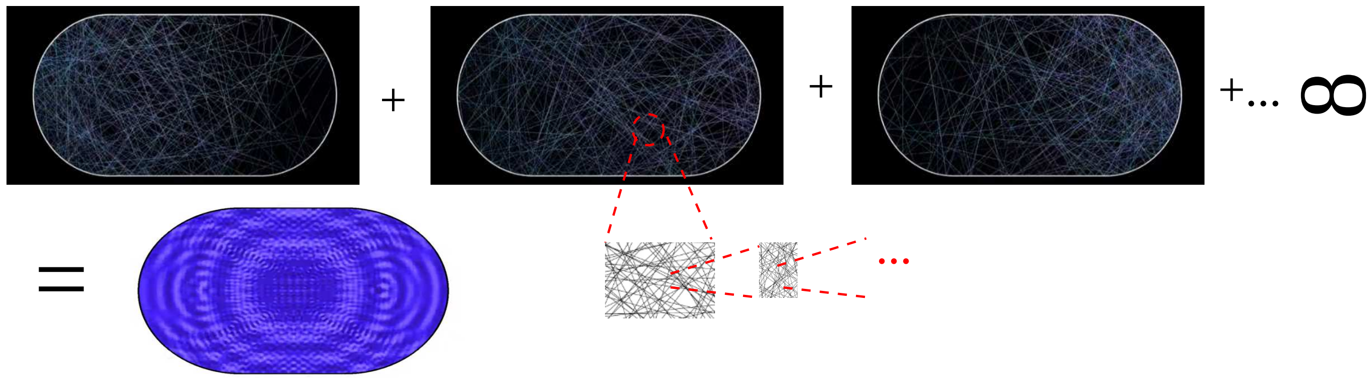
$$\frac{p^\mu}{m}\partial_\mu f(x,p) = 0 \to f(x,p) = f\left(x_0 + \frac{p}{m}t, p\right)$$

obvious solution is just to propagate

What is weird is that "hydro-like" solution possible too (eg vortices)!

$$f(x,p) \sim \exp\left[-\beta_\mu p^\mu\right] \quad , \quad \partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0$$

But obviously unphysical, no force! What's up?



This paradox is resolved by remembering that $f(x,p)$ is defined in an ensemble average limit where the number of particles is not just “large” but uncountable . curvature from continuity!

BUt this suggests Boltzmann equation disconnected from any finite number of particles!

What if $e^{-\beta_\mu p^\mu}$ used to sample strongly coupled particles in ”many finite events”? Thermal fluctuations,Vlasov correlations and Boltzmann scattering ”mix these words”. Many ways to mix,some wrong! What is appropriate?

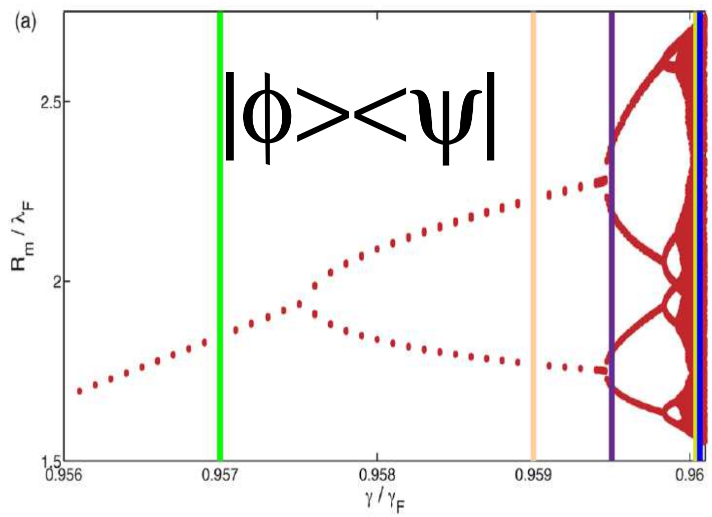
How "different events" correlated is crucial
Villani , https://www.youtube.com/watch?v=ZRPT1Hzze44

Vlasov equation contains all <u>classical</u> correlations. Relativistically numer of particles varies in each event but "evolves" deterministically. but instability-ridden, "filaments", cascade in scales.
$N_{DOF} \to \infty$ invalidates KAM theorem stability

Boltzmann equation "Semi-Classical UV-completion" ov Vlasov equation, first term in BBGK hyerarchy, written in terms of Wigner functions.

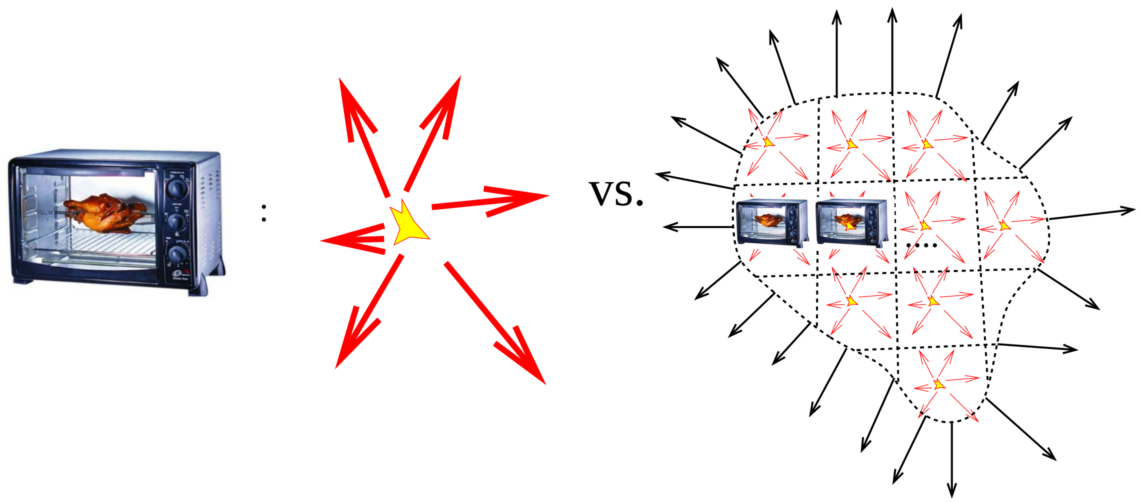
Infinitely unstable jerks on infinitely small scales Random scattering "Both" thermal fluctuations and scattering "mix worlds". How you take limit really important!

Statistical behavior is actually <u>not</u> surprising



Berry/Bohigas/Eigenstate thermalization hypothesys: $\underline{E_{n>>1}}$ of quantum systems whose classical correspondent is <u>chaotic</u> have density matrices that look like pseudo-random. If off-diagonal elements oscillate <u>fast</u> or observables <u>simple</u>, <u>indistinguishable</u> from Micro-canonical ensemble!

What we lack...



We do have mechanisms for “instant thermalization“ (Eigenstate thermalization hypothesis,...) where a state ”looks thermal”. We need to build a hydrodynamics from such a picture away from the many particle limit So fluctuations are included

Perhaps even related to everyday physics?

The
Brazil
nut effect

A proposal for a different point of view: Inverse ("Bayesian") attractor

Close to local equilibrium is not on gradient expansion but the approximate applicability of fluctuation-dissipation
These are not automatically the same!

For smaller fluctuating systems many equivalent definitions of $T_0^{\mu\nu}, \Pi^{\mu\nu}$

Different Boltzmannian entropy but all counted as Gibbsian entropy

If many equivalent choices of $\Pi_{\mu\nu}$ likely in one its "small"! Ideal hydro behavior.

So indeed Ambiguity from fluctuations makes system look like a fluid.

Every statistical theory needs a "state space" and an "evolution dynamics"
The ingredients

State space:Zubarev hydrodynamics Mixes micro and macro DoFs

Dynamics: Crooks fluctuation theorem provides the dynamics via a definition of $\Pi_{\mu\nu}$ from fluctuations

$\hat{T}^{\mu\nu}$ is an operator, so any decomposition, such as $\hat{T}_0^{\mu\nu} + \hat{\Pi}^{\mu\nu}$ must be too!

Zubarev partition function for local equilibrium: think of Eigenstate thermalization...

Let us generalize the GC ensemble to a co-moving frame $E/T \to \beta_\mu T^\mu_\nu$

$$\hat{\rho}(T_0^{\mu\nu}(x), \Sigma_\mu, \beta_\mu) = \frac{1}{Z(\Sigma_\mu, \beta_\mu)} \exp\left[- \int_{\Sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}_0^{\mu\nu} \right]$$

Z is a partition function with a field of Lagrange multiplies β_μ , with microscopic and quantum fluctuations included.

Effective action from $\ln[Z]$. Correction to Lagrangian picture?

All normalizations diverge but hey, it's QFT! (Later we resolve this!)

This is perfect global equilibrium. What about imperfect local?

- Two vectors, $d\Sigma_\mu u_\mu T_0^{\mu\nu}$ $d\Sigma_\mu$ foliation choice not clear (with vorticity it can't be parallel to flow everywhere). Physics should be choice independent. If $d\Sigma_\mu$ close to β_μ , $d\Sigma_\mu$ non-inertial

- Dynamics is not clear. Naively partition function can not depend on time (Adiabatically wrt microscopic scale however it could!) Becattini et al, 1902.01089: Gradient expansion in β_μ . Reproduces Euler and Navier-Stokes, but...
  - 2nd order Gradient expansion (Navier stokes) non-causal perhaps...
  - Use Israel-Stewart, $\Pi_{\mu\nu}$ arbitrary perhaps...
  - Foliation $d\Sigma_\mu$ arbitrary but not clear how to link to Arbitrary $\Pi_{\mu\nu}$

- What about fluctuations? Coarse-graining and fluctuations mix? How does one truncate?

An operator formulation $\hat{T}^{\mu\nu} = \hat{T}_0^{\mu\nu} + \hat{\Pi}_{\mu\nu}$
and $\hat{T}_0^{\mu\nu}$ truly in equilibrium! Each microscopic particle “does not know“ if it ”belongs” to $\hat{T}_0^{\mu\nu}, \hat{\Pi}_{\mu\nu}$

$$\hat{\rho}(T_0^{\mu\nu}(x), \Sigma_\mu, \beta_\mu) = \frac{1}{Z(\Sigma_\mu, \beta_\mu)} \exp\left[- \int_{\Sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}_0^{\mu\nu} \right]$$

describes all cumulants and probabilities

$$\langle T_0^{\mu\nu}(x_1) T_0^{\mu\nu}(x_2) ... T_0^{\mu\nu}(x_n) \rangle = \prod_i \frac{\delta^n}{\delta \beta_\mu(x_i)} \ln Z$$

Equilibrium at ”probabilistic” level and KMS Condition obeyed by ”part of density matrix” in equilibrium, “expand” around that! An operator constrained by KMS condition is still an operator! $\equiv$ time dependence in interaction picture

Does this make sense? Nishioka, 1801.10352 $\langle x|\,\rho\,|x'\rangle =$

$$= \frac{1}{Z}\int_{\tau=-\infty}^{\tau=\infty}\int \left[\mathcal{D}\phi, \mathcal{D}y(\tau)\,\mathcal{D}y'(\tau)\right]\, e^{-iS(\phi y, y')}. \underbrace{\delta\left[y(0^+)-x'\right]\,\delta\left[y'(0^-)-x\right]}_{\frac{\delta J_i(y(0^+))}{\delta J_i(x')}\frac{\delta J_j(y(0^-))}{\delta J_j(x)}}$$

$$\Rightarrow \frac{\delta^2}{\delta J_i(x)\delta J_j(x')}\ln\left[Z_{T_0}(T^{\mu\nu}, J)\times Z_\Pi(J)\right]_{J=J_1(x)+J_2(x')}$$

$J_1(x)+J_2(x')$ chosen to respect Matsubara conditions!

Any ρ can be separated like this for <u>any</u> β_μ . The question is, is this a good approximation? "Close enough to equilibrium"

The source J related to the smearing in "weak solutions". Pure maths angle?

Entropy/Deviations from equilibrium

$$n^{\nu}\partial_{\nu}\left(su^{\mu}\right) = n^{\mu}\frac{\Pi^{\alpha\beta}}{T}\partial_{\alpha}u_{\beta} \quad , \quad \geq 0$$

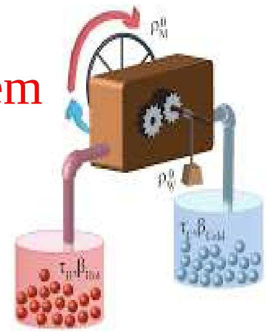
- If n_{μ} arbitrary cannot be true for "any" choice
- 2nd law is true for "averages" anyways, <u>sometimes</u> entropy can decrease

We need a fluctuating formulation!

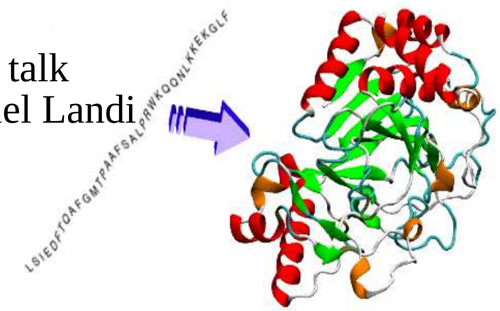
- "Statistical" (probability depends on "local microstates")
- Dynamics with fluctuations, time evolution of β_{μ} distribution

Crooks fluctuation theorem

$P(W)/P(-W)=e^{\Delta s}$



From talk
Gabriel Landi



Relates fluctuations, entropy in <u>small</u> <u>fluctuating</u> systems (Nano,proteins)

P(W) Probability system doing work in its usual thermal evolution

P(-W) Probability of the same system "running in reverse" and decreasing entropy due to a <u>thermal fluctuation</u>

ΔS Entropy produced by $P(W)$

Looks obvious but...

Is valid for systems very far from equilibrium (nano-machines, protein folding and so on)

Proven for Markovian processes and fluctuating systems in contact with thermal bath

Leads to irreducible fluctuation/dissipation: TUR (more later!)

Applying it to locally equilibrium systems within Zubarev's formalism is straight-forward . Since ratios of probabilities, divergences are resolved!

How is Crooks theorem useful for what we did? Guarnieri et al, arXiv:1901.10428 (PRX) derive Thermodynamic uncertainity relations from

$$\hat{\rho}_{ness} \simeq \hat{\rho}_{les}(\lambda) e^{\hat{\Sigma}} \frac{Z_{les}}{Z_{ness}} \quad , \quad \hat{\rho}_{les} = \frac{1}{Z_{les}} \exp\left[-\frac{\hat{H}}{T} \right]$$

$\hat{\rho}_{les}$ is Zubarev operator while Σ is calculated with a Kubo-like formula

$$\hat{\Sigma} = \delta_\beta \Delta \hat{H}_+ \quad , \quad \hat{H}_+ = \lim_{\epsilon \to 0^+} \epsilon \int dt e^{\epsilon t} e^{-\hat{H} t} \Delta \hat{H} e^{\hat{H} t}$$

Relies on

$$\lim_{w \to 0} \left\langle \left[\hat{\Sigma}, \hat{H} \right] \right\rangle \to 0 \equiv \lim_{t \to \infty} \left\langle \left[\hat{\Sigma(t)}, \hat{H}(0) \right] \right\rangle \to 0$$

This "infinite" is "small" w.r.t. hydro gradients. $\equiv$ Markovian as in Hydro with $l_{mfp} \to \partial$ but with operators$\to$ carries all fluctuations with it!

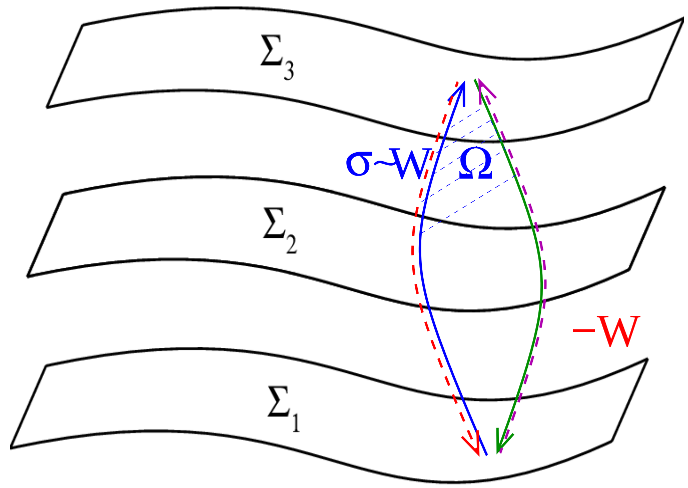
$$P(W)/P(-W) = \exp[\Delta S] \quad \text{Vs} \quad S_{eff} = \ln Z$$

KMS condition reduces the functional integral to a Metropolis type weighting, $\equiv$ periodic time at rest with β_μ

Markovian systems exhibit Crooks theorem, two adjacent cells interaction outcome probability proportional to number of ways of reaching outcome . The normalization divergence is resolved since ratios of probabilities are used . "instant decoherence/thermalization" within each step

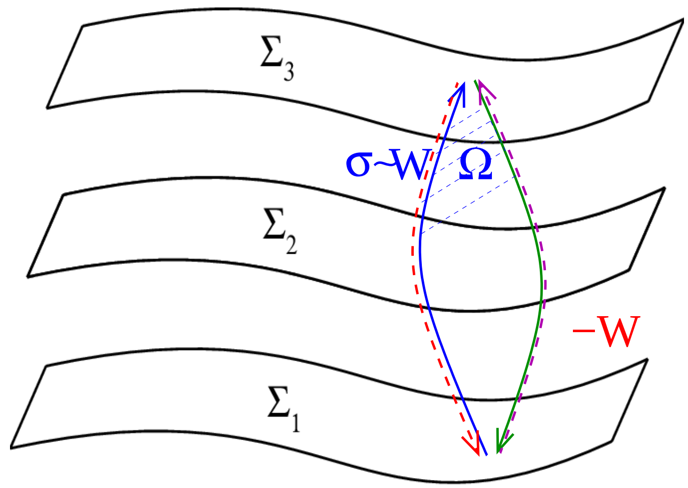
Relationship to gradient expansion similar to relationship between Wilson loop coarse-graining (Jarzynski's theorem, used on lattice ,Caselle et al, 1604.05544) with hadronic EFTs

Applying Crooks theorem to Zubarev hydrodynamics: Stokes theorem



$$-\int_{\Sigma(\tau_0)} \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu\right) = -\int_{\Sigma(\tau')} \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu\right) + \int_\Omega \mathrm{d}\Omega \left(\widehat{T}^{\mu\nu}\nabla_\mu\beta_\nu\right),$$

true for “any” fluctuating configuration.

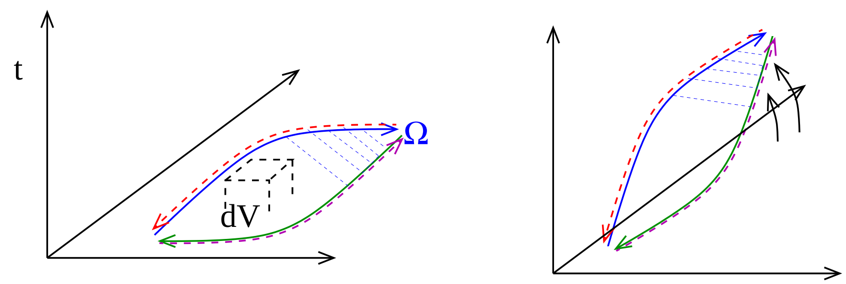


Let us now invert one foliation so it goes "backwards in time" <u>assuming</u> Crooks theorem means

$$\frac{\exp\left[-\int_{\sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}^{\mu\nu}\right]}{\exp\left[-\int_{-\sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}^{\mu\nu}\right]} = \exp\left[\frac{1}{2}\int_{\Omega} d\Omega_\mu^\mu \left[\frac{\hat{\Pi}^{\alpha\beta}}{T}\right] \partial_\beta \beta_\alpha\right]$$

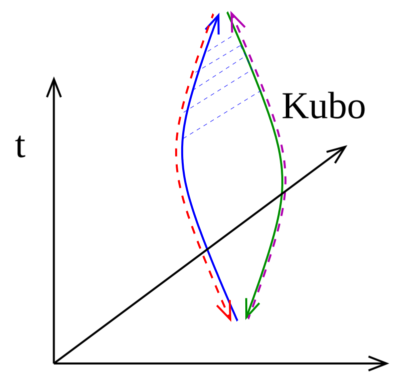
Small loop limit $\left\langle \exp\left[\oint d\Sigma_\mu \omega^{\mu\nu}\beta^\alpha \hat{T}_{\alpha\nu}\right]\right\rangle = \left\langle \exp\left[\int \frac{1}{2} d\Sigma_\mu \beta^\mu \hat{\Pi}^{\alpha\beta}\partial_\alpha\beta_\beta\right]\right\rangle$
A non-perturbative operator equation,divergences cancel out...

$$\left.\frac{\hat{\Pi}^{\mu\nu}}{T}\right|_\sigma = \left(\frac{1}{\partial_\mu\beta_\nu}\right)\frac{\delta}{\delta\sigma}\left[\int_{\sigma(\tau)} d\Sigma_\mu\beta_\nu\hat{T}^{\mu\nu} - \int_{-\sigma(\tau)} d\Sigma_\mu\beta_\nu\hat{T}^{\mu\nu}\right]$$



A sanity check: For a an equilibrium spacelike $d\Sigma_\mu = (dV, \vec{0})$ (left-panel) we recover Boltzmann's $\Pi^{\mu\nu} \Rightarrow \Delta S = \frac{dQ}{T} = \ln\left(\frac{N_1}{N_2}\right)$, for an analytically continued "tilted" panel, Kubo's formula

A sanity check



When $\eta \to 0$ *and* $s^{-1/3} \to 0$ (the first two terms in the hierarchy), Crooks fluctuation theorem gives $P(W) \to 1$ $P(-W) \to 0$ $\Delta S \to \infty$ so Crooks theorem reduces to δ-functions of the entropy current

$$\delta\left(d\Sigma_\mu\left(su^\mu\right)\right) \Rightarrow n^\mu \partial_\mu\left(su^\mu\right) = 0$$

We therefore recover conservation equations for the entropy current, a.k.a. ideal hydro

In summary

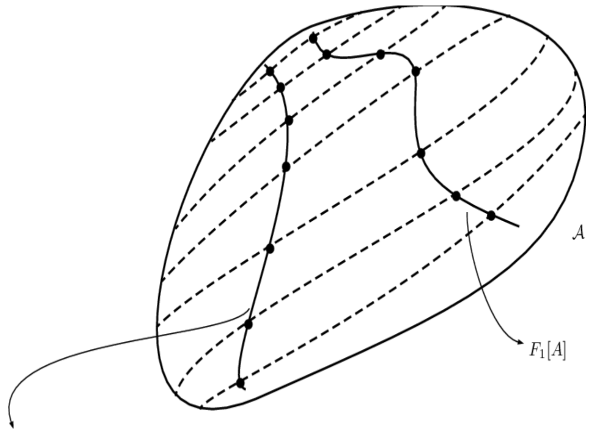
- All known limits are reproduced
- All fluctuations are included
- There is a algorithm to "solve" $\ln \mathcal{Z}$ at each point of time on the lattice.
- but
  - Since $s \sim T \ln \mathcal{Z}$ and fluctuation dynamics driven by entropy differences, semiclassically dynamics independent of $d\Sigma_\mu$. Is it exactly? not sure
  - Algorithm probably too time-consuming to be realistic. Any qualitative conclusions?

So could fluctuations help thermalize? A key insight is redundances

Some qualitative developments: $T_0^{\mu\nu}, \Pi^{\mu\nu}, u^\mu$ are not actually experimental observables! Only total energy momentum tensor

$$\hat{T}^{\mu\nu} = \hat{T}_0^{\mu\nu} + \hat{\Pi}^{\mu\nu}$$

and its correlators are! Changing $d\Sigma_\mu$ in Zubarev $\equiv$ changing $\Pi^{\mu\nu}, T_0^{\mu\nu}$!



Analogy to choosing a gauge in gauge theory?

This is relevant for current hydrodynamic research

Causal relativistic hydrodynamics still contentious, with many definitions

Israel-Stewart Relaxing $\Pi_{\mu\nu}$.

Causal, but up to 9 additional DoFs (not counting conserved charges), blow-up possible (M.Disconzi, 2008.03841). $\Pi_{\mu\nu}$ "evolving" microstates!

BDNK,earlier Hiscock,Lindblom,Geroch,... $\Pi_{\mu\nu} \sim \partial u$ At a price

- Arbitrary (up to causality constaints) u_μ .
- Entropy "temporarily decreases" with perturbations (Gavassino et al, arXiv:2006.09843). Kovtun in 2112.14042 derives BDNK from a truncation of the Boltzmann equation generally violating the H-theorem

For phenomenology because of conservation laws "any" $\partial_\mu T^{\mu\nu}$ "can be integrated" but lack of link with equilibration and multiple definitions of "near-equilibrium" problematic.

If you care about statistical mechanics, price is steep!
"special" time foliation from ergodic hypothesis/Poncaire cycles!

But entropy decrease physically reasonable from Zubarev definition. But not from H-theorem!

Fluctuations come with redundances in $T_0^{\mu\nu}, \Pi^{\mu\nu}$

Could these definitions of u_μ be just "Gauge" choices?

What is a gauge theory,exactly?

$$\mathcal{Z}=\int \mathcal{D}A^{\mu}\exp\left[S[F_{\mu\nu}\right]\equiv\int \mathcal{D}A_1^{\mu}\mathcal{D}A_2^{\mu}\exp\left[S[A_1^{\mu}\right]$$

$A^{\mu}_{1,2}$ can be separated since physics sensitive to derivatives of $\ln\mathcal{Z}$

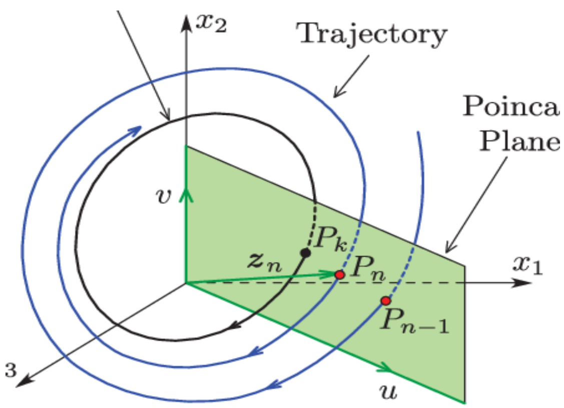
$$\ln\mathcal{Z}=\Lambda+\ln\mathcal{Z}_G\quad,\quad Z_G=\int\mathcal{D}\mathcal{A}^{\mu}\delta\left(G(A^{\mu})\right)\exp\left[S(A_{\mu})\right]$$

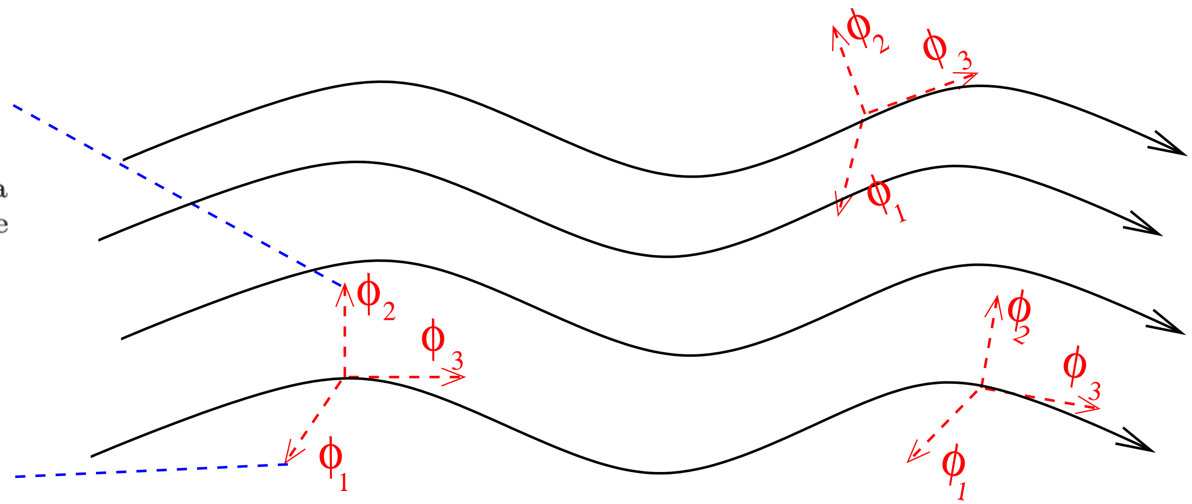
Ghosts come from expanding $\delta(...)$ term. In Zubarev

$$Z=\int\mathcal{D}\phi\quad,\quad "S"=d\Sigma_{\nu}\beta_{\mu}T^{\mu\nu}$$

Multiple $T_{\mu\nu}(\phi)\rightarrow$ Gauge-like configuration . Related to Phase space fluctuations of ϕ

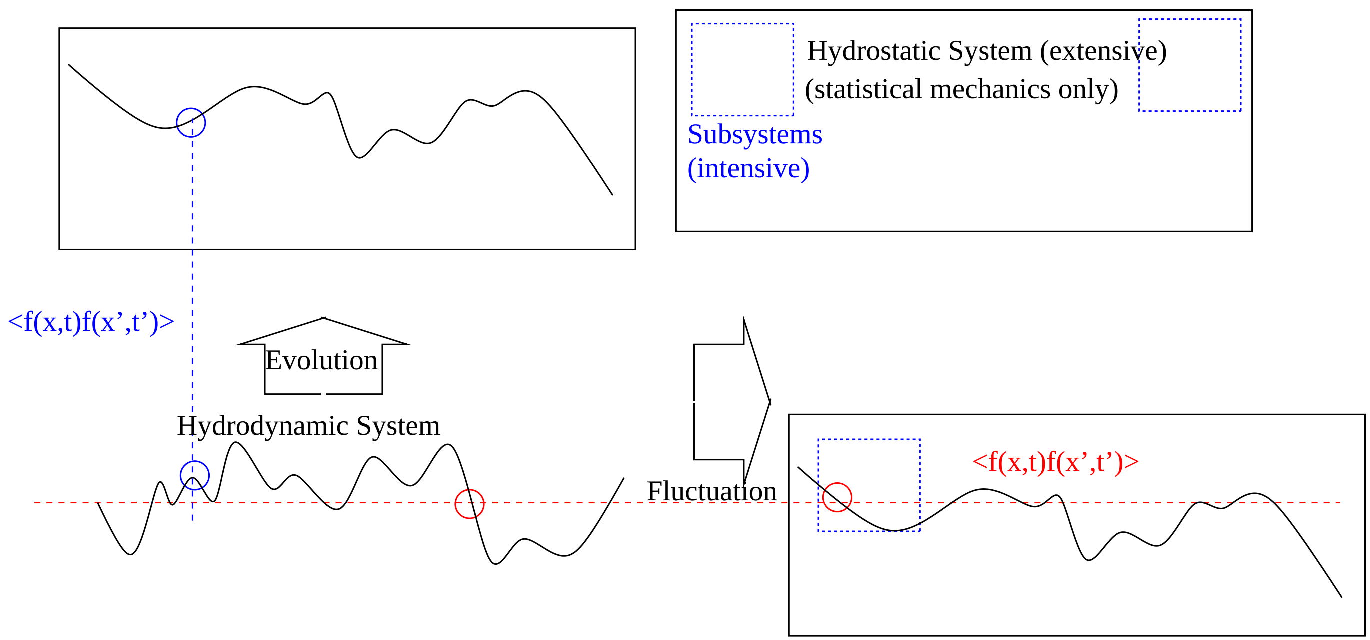
How to make physics fully "gauge"-invariant? Ergodicity/Poncaire cycles meet relativity slightly away from equilibrium!



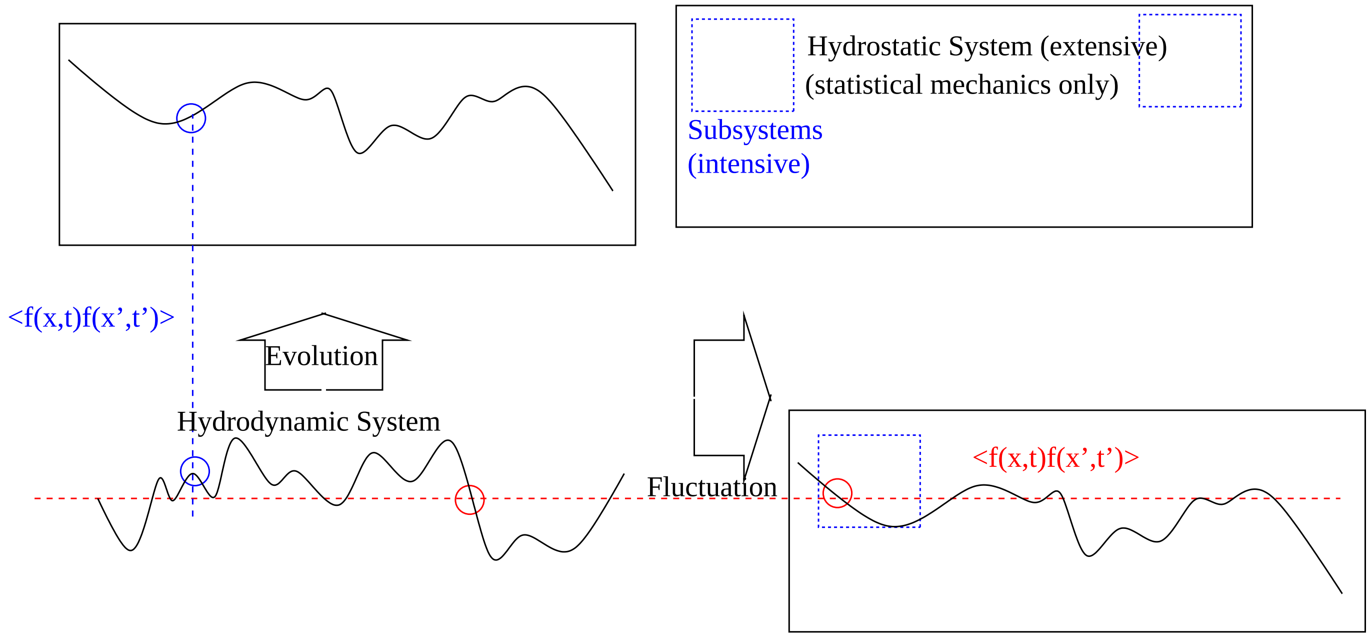


Gibbs entropy level+relativity : Lack of equilibrium is equivalent to "loss of phase" of Poncaire cycles. one can see a slightly out of equilibrium cell <u>either</u> as a "mismatched u_μ" (fluctuation) or as lack of genuine equilibrium (dissipation)

How to make physics fully “gauge”-invariant?



Fluctuation-dissipation at the cell level could do it! We don’t know if a "step" is fluctuation ($T_0^{\mu\nu}$ or evolution ($\Pi_{\mu\nu}$)-driven!



But in hydro $T_0^{\mu\nu}, \Pi_{\mu\nu}$ treated very differently! "Sound-wave" $u \sim \exp[ik_\mu x^\mu]$ or "non-hydrodynamic Israel-Stewart mode?"
$D\Pi_{\mu\nu} + \Pi_{\mu\nu} = \partial u$
Only in EFT $1/T \ll l_{mfp}$ they are truly different!

Infinitesimal transformation $dM_{\mu\nu}$ such that $dM_{\mu\nu}(x)\frac{\delta \ln \mathcal{Z}_E[\beta_\mu]}{dg^{\alpha\mu}(x)} = 0$

Change in microscopic fluctuation $\ln \mathcal{Z} \rightarrow \ln \mathcal{Z} + d \ln \mathcal{Z}$

$$d \ln \mathcal{Z} = \sum_{N=0}^{\infty} \int \prod_{j=1}^{N} d^4 p_j \delta \left(E_N(p_1, ...p_j) - \sum_j p_j^0 \right) \sqrt{|dM|} \exp \left(-\frac{dM_{0\mu} p^\mu}{T} \right)$$

Change in macroscopic dissipative term

$$\Pi_{\mu\nu} \rightarrow \Pi_{\alpha\gamma} \left(g_\mu^\alpha g_\nu^\gamma - g_\mu^\alpha dM_\nu^\gamma - g_\nu^\gamma dM_\mu^\alpha \right) \quad , \quad u_\mu \rightarrow u_\alpha \left(g_\mu^\alpha - dM_\mu^\alpha \right)$$

For $1/T \ll l_{mfp}$ probability$\rightarrow 0$, $1/T \sim l_{mfp}$ many "similar" probabilities!

The "gauge-symmetry" in practice

Generally $dM_{\mu\nu} = \Lambda^{-1}_{\alpha\mu} dU^{\alpha\beta} \Lambda_{\beta\mu}$

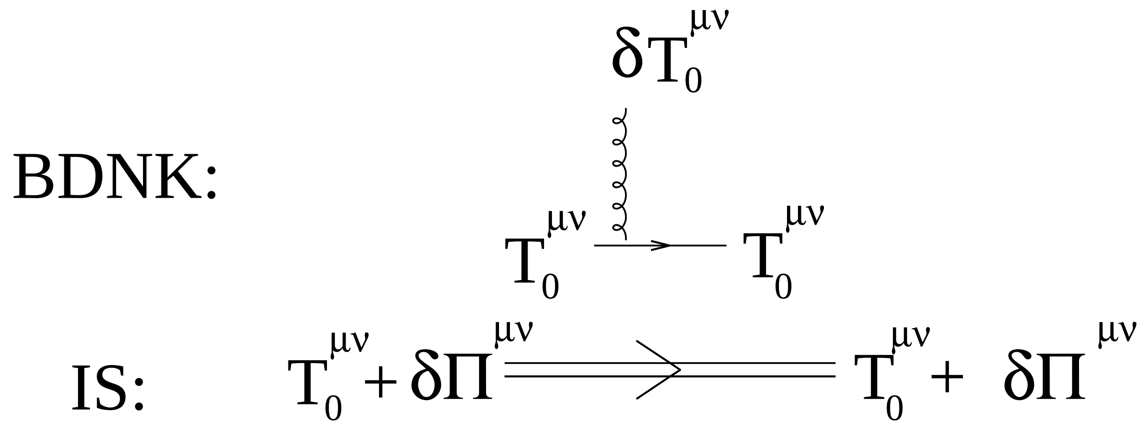
$$d\left[\ln \Pi_{\alpha\beta}\right] \Lambda^{\alpha\mu} \left(\Lambda^{\beta\nu}\right)^{-1} = \eta^{\mu\nu} d\mathcal{A} + \sum_{I=1,3} \left(d\alpha_I \hat{J}_I^{\mu\nu} + d\beta_I \hat{K}_I^{\mu\nu} \right)$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

which move components from $\Pi_{\mu\nu}$ to Q_μ as well as $K_{1,2,3}$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

An example... bulk viscosity



$$e_{IS} \to e + (e+p)\tau \frac{\dot{e}}{e+p} + \left((e+p)\tau + \frac{c_V}{s}\zeta \right) \partial_\mu u^\mu \quad , \quad p_{IS} \to p + \Pi$$

Considering c_V controls energy fluctuations, shift from IS to BDNK equivalent to relabeling Π dynamics as interaction with a fluctuation-generated sound wave.

Towards hydrodynamic Gibbsian entropy definition !

$$\int \mathcal{D}\phi e^{-S(\phi)} \underbrace{\rightarrow}_{coarse-grain} \int \mathcal{D}\alpha_{I=1,2,3}\mathcal{D}\beta_{I=1,2,3}\ \mathcal{D}\left[\mathcal{A}, e, p, u_\mu, \Pi_{\mu\nu}\right]$$

$$\delta\left(M_{\alpha\beta}\left[\mathcal{A}, \alpha_I, \beta_I\right] T^{\alpha\mu}\right)$$

rotate "Gradient expansion" in $1/T, l_{mfp}$ parameter space.
Away from Boltzmann equation regime, $f(x,p) \rightarrow$ Functional

lagrangian , $\ln \mathcal{Z}$ subject to $\delta(...)$ constraint.

Causality also defined via correlator $[T_{\mu\nu}(x), T_{\mu\nu}(x')]$ $e, u_\mu \Pi_{\mu\nu}$ could be non-causal!

An "improved" Zubarev picture

$$\mathcal{Z}(\tau,\Sigma,\beta) = \int \mathcal{D}\phi \exp\left[- \int d\Sigma_0(\tau) \left(\underbrace{\beta_\nu \hat{T}_0^{0\nu}}_{"Equilibrium"} + \underbrace{\beta^0 \hat{\Pi}^{\alpha\beta} \partial_\alpha \beta_\beta}_{"rest"} \right) \right]$$

Many choices, $\ln \mathcal{Z}\,(\tau) \to \ln \mathcal{Z}\,(\tau + d\tau)$ should be independent of Σ_μ Both $\hat{T}_0^{\mu\nu}, \hat{\Pi}^{\mu\nu}$ operators,fluctuate but only sum observable

$$\Sigma \to \Sigma' \quad , \quad \ln \mathcal{Z}(\Sigma) \equiv \ln \mathcal{Z}(\Sigma') \quad , \quad \beta_\mu, \Pi_{\mu\nu} \to \beta'_\mu, \Pi'_{\mu\nu}$$

Evolution "flow" of $\beta'_\mu, \Pi'_{\mu\nu}$ under deformation of Σ_μ

$$\ln \mathcal{Z}(\Sigma,\beta) \equiv \ln \mathcal{Z}(\Sigma',\beta'), \qquad , \qquad \langle T_0^{\mu\nu} + \Pi_{\mu\nu}\rangle_\Sigma = \langle T_0^{\mu\nu} + \Pi_{\mu\nu}\rangle_{\Sigma'} = \frac{\delta}{\delta g_{\mu\nu}} \ln \mathcal{Z}$$

Constraint of the form $F\left(\beta, \partial\beta, \left\langle \beta^2 \right\rangle\right) = 0$ similar to DSE. Perhaps via the Gravitational Ward identity

$$\partial^\alpha \left\{ \left\langle \left[\hat{T}_{\mu\nu}(x), \hat{T}_{\alpha\beta}(x') \right] \right\rangle - \right.$$

$$\left. - \, \delta(x-x') \left(g_{\beta\mu} \left\langle \hat{T}_{\alpha\nu}(x') \right\rangle + g_{\beta\nu} \left\langle \hat{T}_{\alpha\mu}(x') \right\rangle - g_{\beta\alpha} \left\langle \hat{T}_{\mu\nu}(x') \right\rangle \right) \right\} = 0$$

Characterizing these gauge redundancies

Grossi,Floerchinger, 2102.11098 (PRD) Let us define a J co-moving with u_μ and use the "exact" (before coarse-graining) partition function to build

$$\Gamma(\phi) = \mathrm{Sup_J}\left(\int J(x)\phi(x) - i \ln \mathcal{Z}[J] \right)$$

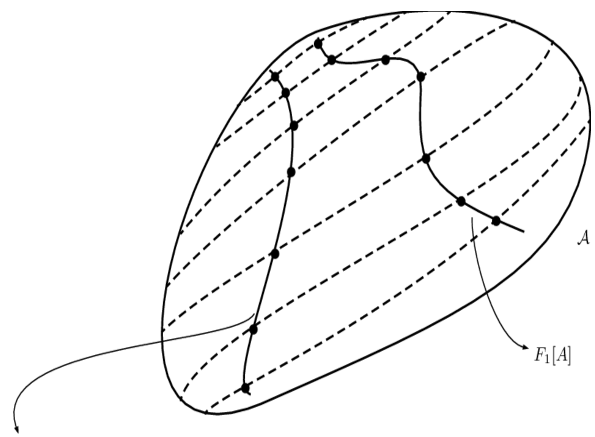
$u_\mu \to u'_\mu$ non-inertial and does not change $\langle T_{\mu\nu} \rangle$, so one can define

$$J_{\mu\nu\gamma} = \frac{1}{\sqrt{g}} \frac{\delta \ln \mathcal{Z}[J']}{\delta \Gamma^{\alpha\nu\gamma}} \quad , \quad D_\mu J^{\mu\nu\gamma} = 0$$

Setting the gauge at the level of the microscopic approximately thermalized partition function equivalent adding auxiliary field $D_\mu M_{\alpha\beta} = 0$ to

$$\mathcal{Z}[J_{\alpha\beta\gamma}] = \int \mathcal{D}\phi \mathcal{D} M_{\alpha\beta} \exp\left[\int det[M] d^4 x \mathcal{L}\left(\phi, \partial_\mu + \Gamma ...\right) + \int d\Sigma^\gamma M^{\alpha\beta} J_{\alpha\beta\gamma} \right]$$

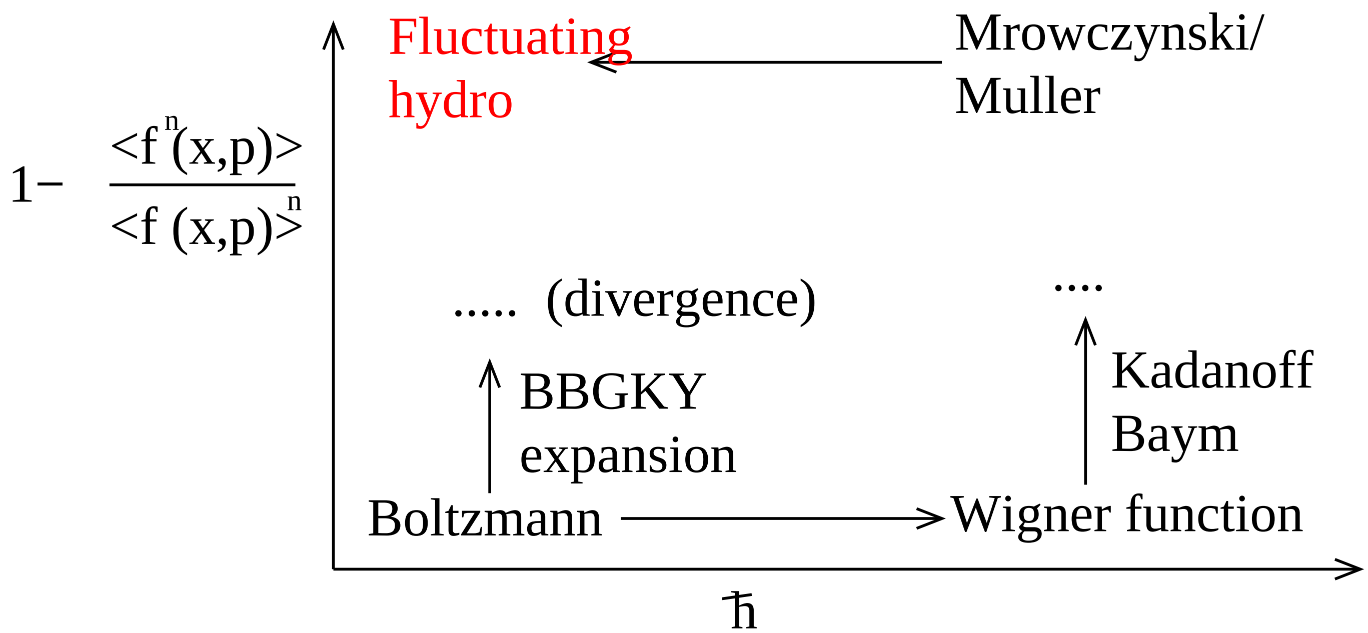
Cool but what about thermalization in small systems?
Initial and final state described by many equivalent trajectories



One of them could be close to an ideal-looking one. "reverse" attractor Few particles with strong interaction (Eigenstate thermalization?) correspond to many hydro like-configurations $\{u_\mu, \Pi_{\mu\nu}\}$ with fluctuations , within same Gibbs entropy class. some closer to ideal? No symmetries necessary!

Irrelevant in everyday liquids since $l_{mfp} \gg 1/T$ or AdS/CFT since $N_c \ll \infty$ but perhaps not for QGP!

Back to transport



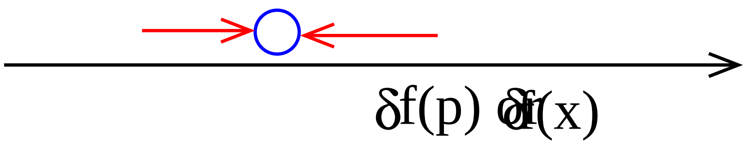
Boltzmann equation emerges as a double limit from microscopic correlations, $\hbar \to 0$ Relaxing the latter limit would destroy statistical independence CHSH relations , so probably not relevant (phases "chaotic"). But fluctuating hydro "non-perturbative" in correlations

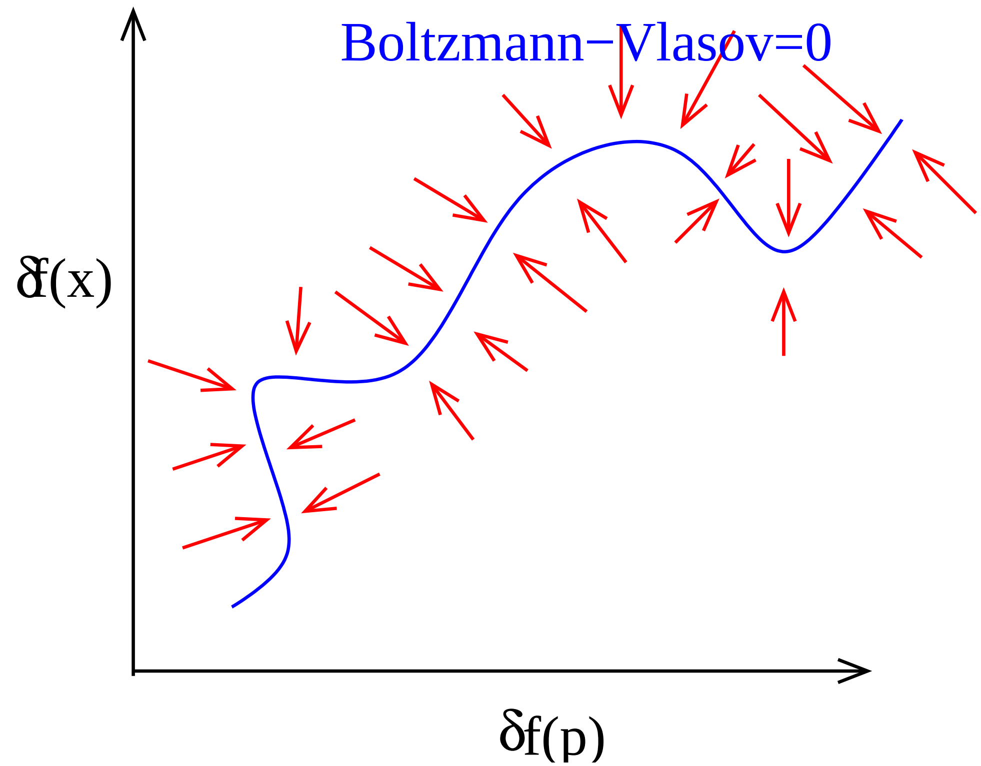
Finite number of particles: $f(x,p)$ not a <u>function</u> but a <u>functional</u> ($\mathcal{F}(f(x,p)) \underbrace{\rightarrow}_{Boltzmann} \delta\left(f' - f(x,p)\right)$), incorporating continuum of functions and <u>all correlations</u>. <u>Perhaps solvable!</u>

$$\frac{p^{\mu}}{\Lambda}\frac{\partial}{\partial x^{\mu}}f(x,p) = \left\langle \underbrace{\hat{C}[\tilde{W}(\tilde{f}_1,\tilde{f}_2)] - g\frac{p^{\mu}}{\Lambda}\hat{F}^{\mu\nu}[\tilde{f}_1,\tilde{f}_2]\frac{\delta}{\delta\tilde{f}_{1,2}}\tilde{W}\left(\tilde{f}_1,\tilde{f}_2\right)}_{How\quad many\quad A-B=0?} \right\rangle$$

Wigner functional to $\mathcal{O}\left(h^0\right)$. What is the effect? If only Boltzmann term not much!

If Both Vlasov and Boltzmann terms, redundancy-ridden!





One can deform $f(x,p)$ by $\delta f(x)$ or $\delta f(p)$ so that $\hat{C} - \hat{W}$ cancels. In ensemble average deformation makes no sense, but away from it it does!

$$f(x,p) \to f'(x,p) \quad , \quad \underbrace{\hat{C}\left(f(x,p), f'(x,p)\right)}_{lim_{f\to f'} \sim \partial f/\partial x} = \underbrace{\hat{V}^{\mu}\left(f(x,p), f'(x,p)\right)\frac{\partial f}{\partial p_{\mu}}}_{lim_{f\to f'} \sim \partial f/\partial p}$$

Infinite number of redundancies! Close to local equilibrium limit...

$$\left\{ \begin{array}{c} f(x,p) \\ f'(x,p) \end{array} \right\} \sim \exp\left[- \left\{ \begin{array}{c} \beta_{\mu}(x,t) \\ \beta'_{\mu}(x,t) \end{array} \right\} p^{\mu} \right] \quad , \quad \lim_{f \to f'} \left\{ \begin{array}{c} \hat{C}[f,f'] \\ \hat{V}[f,f'] \end{array} \right\} \sim \left\{ \begin{array}{c} \langle \partial \beta \rangle \\ \langle \beta^2 \rangle \end{array} \right\}$$

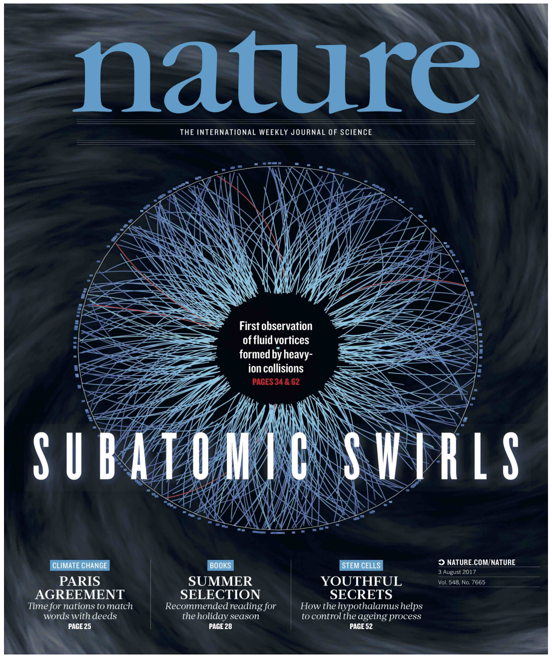
and these redundances look like the hydro ones

PS: transfer of micro to macro DoFs experimentally proven!

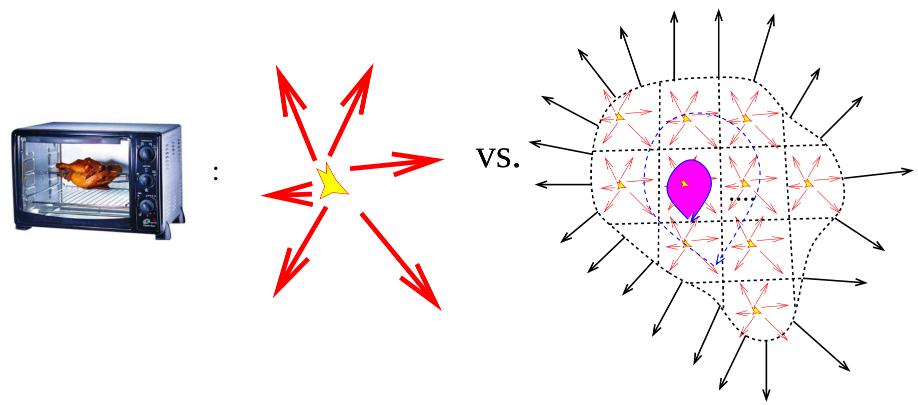
STAR collaboration
1701.06657

NATURE
August 2017

Polarization by vorticity in heavy ion collisions



Could give new talk about this, but will mention hydro with spin not developed and a lot of conceptual debates Pseudo-gauge dependence if both spin and angular momentum present in fluid? Gauge symmetry "ghosts"? GT,1810.12468 (EPJA) . redundances?



Pseudo-gauge symmetries physical interpretation: T.Brauner, 1910.12224

$$x^{\mu} \to x^{\mu} + \epsilon \zeta^{\mu}(x) \quad , \quad \psi_a \to \psi_a + \epsilon \psi_a' \to \mathcal{L} \to \mathcal{L}$$

$\ln \mathcal{Z}$ Invariant, but $\langle O \rangle$ generally is not. Spin $\leftrightarrow$ fluctuation, need equivalent of DSE equations! $D \langle O \rangle = 0 \to D \langle O \rangle = \langle O_I O_J \rangle$

Pseudo-gauge issue suggests spin not a simple coarse-graining ("small vortex $\equiv$ spin"). Need to include fluctuations to restore a pseudo-gauge independent dynamics

Conclusions

- Linking hydrodynamics to statistical mechanics is still an open problem
  Only top-down models (Boltzmann,AdS/CFT) rather than bottom-up theory
  Is hydro universal? what are its limits of applicability? still open question

  The observation of hydro-like behavior in small systems liable to fluctuations makes this explicit!

- Crooks fluctuation theorem could provide such a link!

- redundances play crucial role in fluctuations, could mean small systems achieve "thermalization" quicker! inverse attractor!

- An obvious extension/application is...

SPARE SLIDES

There is more to hydro than the Knudsen number

Power counting:

B.Betz,D.Henkel,D.Risc 0812.1440

3 length scales: 2 microscopic, 1 macroscopic

- thermal wavelength $\lambda_{\rm th} \sim \beta \equiv 1/T$
- mean free path $\ell_{\rm mfp} \sim (\langle\sigma\rangle n)^{-1}$

What if these are ~?

$\langle\sigma\rangle$ averaged cross section, $n \sim T^3 = \beta^{-3} \sim \lambda_{\rm th}^{-3}$

- length scale over which macroscopic fluid fields vary $L_{\rm hydro}$, $\partial_\mu \sim L_{\rm hydro}^{-1}$

Note: since $\eta \sim (\langle\sigma\rangle\lambda_{\rm th})^{-1}$ $\Longrightarrow$ $\frac{\ell_{\rm mfp}}{\lambda_{\rm th}} \sim \frac{1}{\langle\sigma\rangle n}\frac{1}{\lambda_{\rm th}} \sim \frac{\lambda_{\rm th}^3}{\langle\sigma\rangle\lambda_{\rm th}} \sim \frac{\lambda_{\rm th}^3}{\langle\sigma\rangle\lambda_{\rm th}} \sim \frac{\eta}{s}$

s entropy density, $s \sim n \sim T^3 = \beta^{-3} \sim \lambda_{\rm th}^{-3}$

$$\underbrace{l_{micro}}_{\sim s^{-1/3}, n^{-1/3}} \ll \underbrace{l_{mfp}}_{\sim \eta/(sT)} \ll L_{macro}$$

Second inequality was developed so far, but first is suspect! experimentally

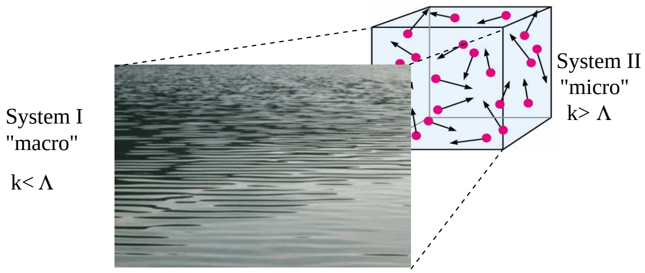
$$\underbrace{l_{micro}}_{\sim s^{-1/3}, n^{-1/3}} \ll \underbrace{l_{mfp}}_{\sim \eta/(sT)} \ll L_{macro}$$

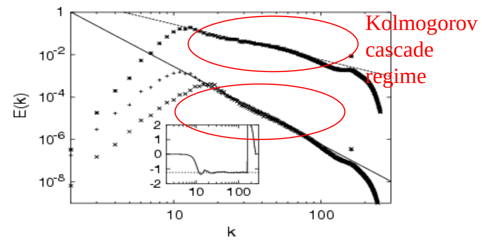
Weakly coupled: Ensemble averaging in Boltzmann equation good up to $\mathcal{O}\left((1/\rho)^{1/3}\partial_\mu f(...)\right)$
Strongly coupled: classical supergravity requires $\lambda \gg 1$ but $\lambda N_c^{-1} = g_{YM} \ll 1$ so

$$\frac{1}{TN_c^{2/3}} \ll \frac{\eta}{sT} \quad \left(\quad or \quad \frac{1}{\sqrt{\lambda}T}\right) \ll L_{macro}$$

QGP: $N_c = 3 \ll \infty$,so $l_{micro} \sim \frac{\eta}{sT}$. Cold atoms: $l_{micro} \sim n^{-1/3} > \frac{\eta}{sT}$?

Why is $l_{micro} \ll l_{mfp}$ necessary? microscopic fluctuations (which have nothing to do with viscosity) will drive fluid evolution. $\Delta\rho/\rho \sim C_V^{-1} \sim N_c^{-2}$



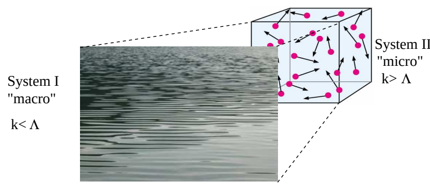


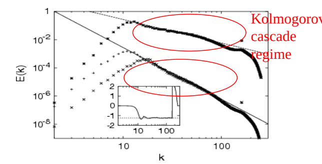
A classical low-viscosity fluid is turbulent. Typically, low-k modes cascade into higher and higher k modes In a non-relativistic incompressible fluid

$$\eta/(sT) \ll L_{eddy} \ll L_{boundary} \quad , \quad E(k) \sim \left(\frac{dE}{dt}\right)^{2/3} k^{-5/3}$$

For a classical ideal fluid, no limit! since $\lim_{\delta\rho\to 0, k\to\infty} \delta E(k) \sim \delta\rho k c_s \to 0$ but quantum $E \geq k$ so energy conservation has to cap cascade.

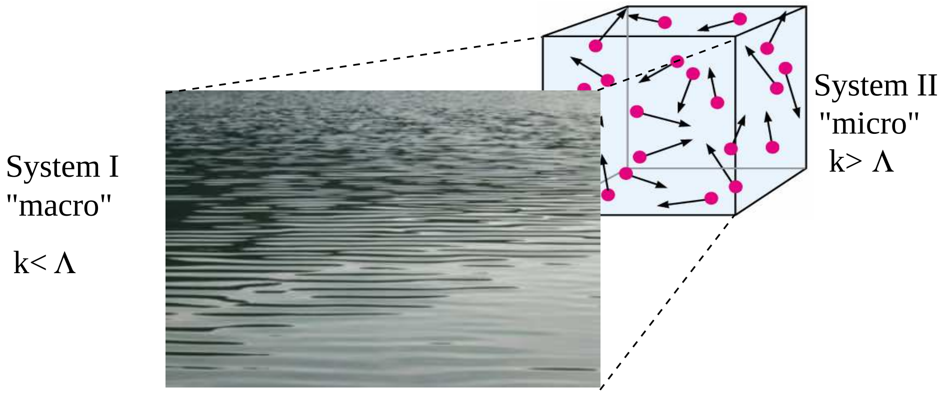
More fundamentally: take stationary slab of fluid at local equilibrium.

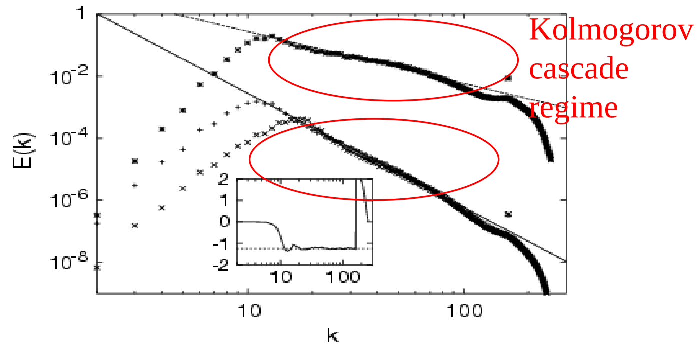




Statistical mechanics: This is a system in global equilibrium, described by a partition function $Z(T, V, \mu)$, whose derivatives give expectation values $\langle E \rangle$,fluctuations $\left\langle (\Delta E)^2 \right\rangle$ etc. in terms of conserved charges. All microstates equally likely, which leads to preferred macrostates!

Fluid dynamics: This is the state of a field in local equilibrium which can be perturbed in an infinity of ways. The perturbations will then interact and dissipate according to the Euler/N-S equations. Smaller η/s , the closer to local equilibrium (SM applies to cell) but the longer the timescale to global equilibrium (SM applies to system).





- Provided state is localized, local equilibrium is ”global equilibrium in every cell”, global equilibrium with spin, forces ”non-local” A.Palermo et al,2007.08249,2106.08340 ”global” equilibrium not necessarily stable against hydro perturbations I <u>think</u> ”real” global equilibrium built up from local equilibria

- Dissipation scale in local equilibrium $\eta/(Ts)$, global equilibration timescale $(Ts)/\eta$ .turbulence drastically changes this ,but ”when does a small perturbation become a microstate?”

Some insight from maths

Millenium problem: existence and smoothness of the Navier-Stokes equations

Important tool are "weak solutions" , similar to what we call "coarse-graining".

$$F\left(\frac{d}{dx}, f(x)\right) = 0 \Rightarrow F\left(\int \frac{d}{dx}\phi(x)..., f(x)\right) = 0$$

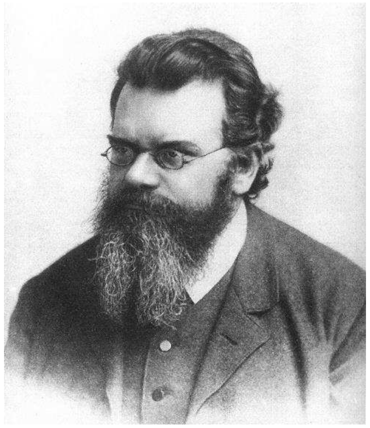
$\phi(x)$ "test function", similar to coarse-graining!

Existance of Wild/Nightmare solutions and non-uniqueness of weak solutions shows this tension is non-trivial, coarse-graining “dangerous”

I am a physicist so I care little about the ”existence of ethernal solutions” to an approximate equation, Turbulent regime and microscopic local equilibria need to be consistent

Thermal fluctuations could both ”stabilize” hydrodynamics and ”accellerate” local thermalization
But where do microstates,”local” microstates fit here?

the battle of the entropies



Boltzmann entropy is usually a property of the "DoF", and is "kinetic" subject to the H-theorem which is really a consequence of the not-so-justified molecular chaos assumption. Gibbsian entropy is the log of the area of phase space, and is justified from coarse-graining and ergodicity, but hard to define it in non-equilibrium. The two are different even in equilibrium, with interactions! Note, Von Neumann $\langle ln\hat{\rho}\rangle$ Gibbsian