Going to the light-front with contour deformations

G. Eichmann[1] **Eduardo Ferreira**[1] A. Stadler[1]

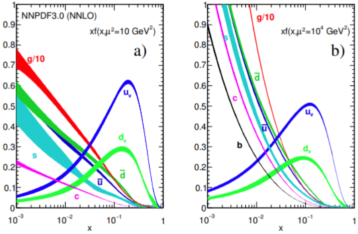
[1]Departamento de Física – Instituto Superior Técnico
LIP – Laboratório de Instrumentação e Física Experimental de Partículas

LIP/IDPASC PhD Student Workshop - 6/7th July 2022

Hadrons on the Light Front

Focus: Hadrons on the light front, $x^+ = 0$.

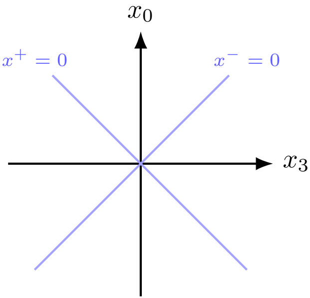
- Natural frame for defining parton distribution functions: PDFs, TMDs, ...

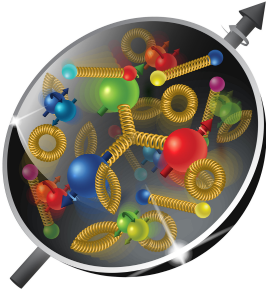


- Future: COMPASS/AMBER @ CERN EIC @ Brookhaven National Laboratory.

(AMBER: arXiv:1808.00848)

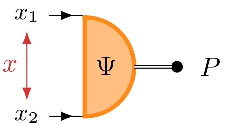
(EIC: Eur. Phys. J. A 52.9 (2016))





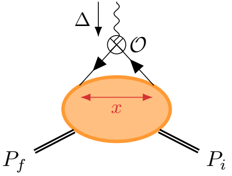
- Bethe-Salpeter Wavefunction

  $\langle 0 | \, \mathrm{T}\Phi(x)\Phi(0) \, | P \rangle$

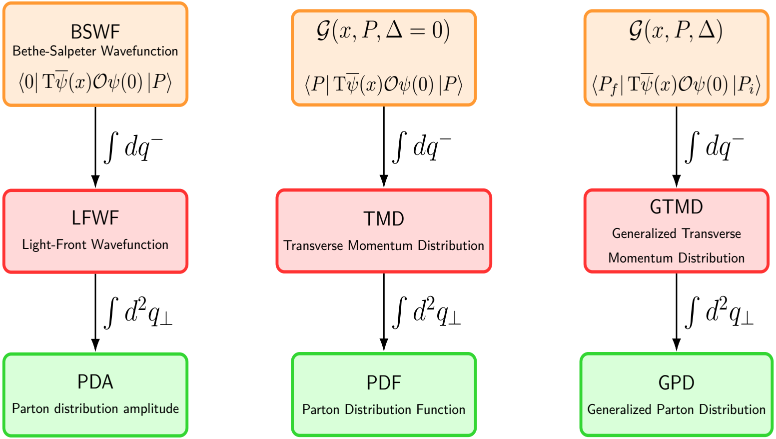


- Generic Correlator

  $\langle P_f | \, \mathrm{T}\Phi(x)\mathcal{O}\Phi(0) \, | P_i \rangle$



- With $x^+ = x^0 + x^3$, $x^- = x^0 - x^3$, $\vec{x}_\perp = \{x^1, x^2\}$.



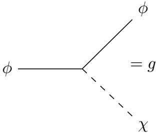
(Lorce, Pasquini, Vanderhaeghen; 2011)

Scalar toy model

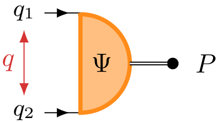
Scalar toy model:

ϕ of mass m

χ of mass μ

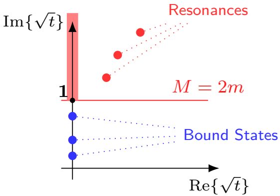


- **Objective:** Calculate BSWF in Euclidean metric for two ϕ particles,



- Need to solve the BSE:

$$\Psi = \mathbf{G_0}\psi \qquad \psi = \mathbf{K}\mathbf{G_0}\psi$$

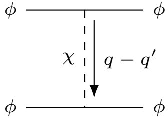


- $\mathbf{G_0}$ — We use tree level propagators

$$\mathbf{G_0} = \frac{1}{q_1^2 + m^2}\frac{1}{q_2^2 + m^2}$$

- $\mathbf{K}$ — We assume a single scalar exchange

$$\mathbf{K} = \frac{g^2}{(q - q')^2 + \mu^2}$$



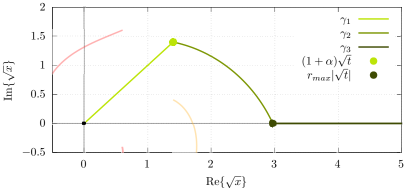
- The BSWF is a function of the kinematic invariants:

$$-M^2 = \frac{P^2}{4m^2} = t \quad \frac{k^2}{m^2} = x \quad \omega = \hat{k} \cdot \hat{P}$$

$$\psi(x, \omega, t, \alpha) = \frac{m^4}{(2\pi)^3}\frac{1}{2}\int_0^\infty dx' x' \int_{-1}^{1} d\omega' \sqrt{1 - \omega'^2}\mathbf{G_0}(x', \omega', t, \alpha)$$
$$\times \int_{-1}^{1} dy\, \mathbf{K}(x, \omega, x', \omega', y)\psi(x', \omega', t, \alpha)$$

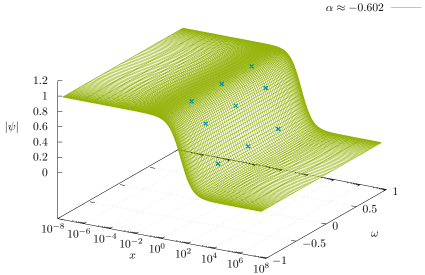
Integration Path

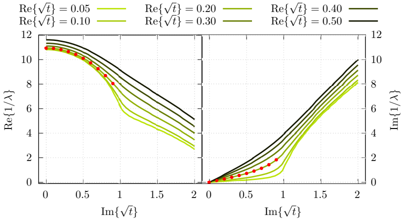
- Constraints:
  1. Must go though $(1+|\alpha|)\sqrt{t}$.
  2. $\mathrm{Re}\{\sqrt{x}\}$ and $|\sqrt{x}|$ must always increase.



- $\boldsymbol{\gamma_1}$ — Line from the origin to $(1+|\alpha|)\sqrt{t}$.
- $\boldsymbol{\gamma_2}$ — Return to the real axis, with increasing radius.
- $\boldsymbol{\gamma_3}$ — $\sqrt{x}\to\infty$ on the real axis.

- If the cuts don't cross the real axis $\implies$ no deformation needed.





(Kusaka, Williams; 1995); (Sauli, Adam, Jr; 2003)
(Karmanov, Carbonell; 2006); (Frederico, Salme, Viviani; 2014)

Light-Front Wavefunction

- The LFWF is defined as:

$$\Psi_{LF}\left(\alpha, k_\perp, P\right) = \mathcal{N}\int dq^- \;\Psi(q,P)\big|_{q^+=\frac{\alpha}{2}P^+,q_\perp=k_\perp}$$

- In our kinematic variables:

$$q^- = -\frac{2m^2}{P^+}\left(2\sqrt{x}\sqrt{t}\omega + \alpha t\right)$$

α and $x = \frac{k^2}{m^2} = \frac{k_\perp^2}{m^2}$ and t are external variables

- Need the BSWF in $\omega \in (-\infty, \infty)$.
- We use the Schlessinger method for analytic continuation:

$$R(\omega) = \cfrac{f(\omega_1)}{1+\cfrac{a_1(\omega-\omega_1)}{1+\cfrac{a_2(\omega-\omega_2)}{1+\cfrac{a_3(\omega-\omega_3)}{\cdots}}}}$$

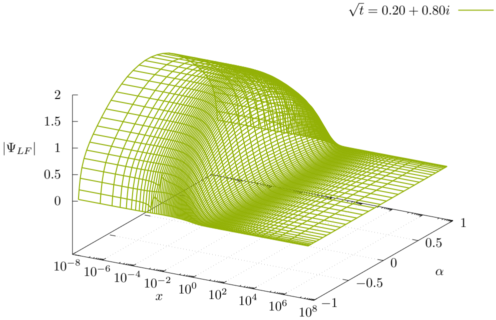
- $\{a_i\}$ obtained by imposing $R(\omega_i) = f(\omega_i)$

Definition of the LFWF

$$\Psi_{LF}(\alpha, x, t) = \mathcal{N}\frac{2\sqrt{x}\sqrt{t}}{i\pi}\int_{-\infty}^{\infty} d\omega\,\Psi(x,\omega,t,\alpha)$$

Results in the light-front

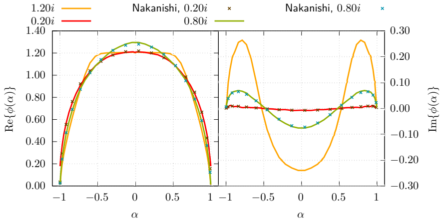
- Results for the LFWF:



  - Symmetric in α.
  - Vanishes at $\alpha = \pm 1$.

- Parton distribution amplitude: in our variables

$$\phi(\alpha) = \int_0^\infty dx\, \Psi_{LF}(x, \alpha, t)$$



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- More details can be found in:

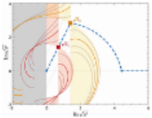
Editors' Suggestion

PDF HTML

Going to the light front with contour deformations

Gernot Eichmann, Eduardo Ferreira, and Alfred Stadler

Phys. Rev. D **105**, 034009 (2022) – Published 10 February 2022



This work introduces a new method for computing correlation functions on the light front. Starting from Bethe-Salpeter equations, the authors present calculations using contour deformation and analytic continuation, finding good agreement with established methods. This approach shows promise as an efficient and generalizable technique with applications to parton distribution functions and related constructions.

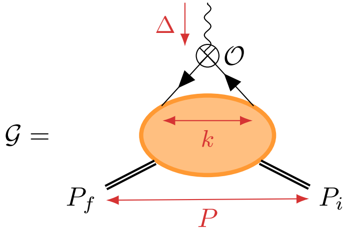
Show Abstract +

G. Eichmann, EF, A. Stadler; Phys. Rev. D 105, 034009 (2022)

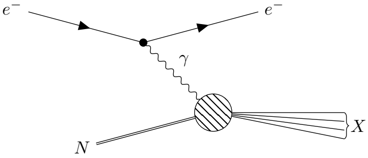
Hadronic Matrix Elements

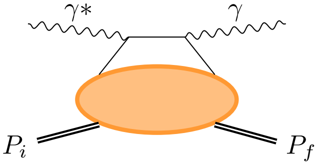
- **Focus:** Understading hadronic composition and interactions:

$$\mathcal{G}^{\mu\nu\rho\dots}_{\alpha\beta\delta\dots} = \langle P_f |\, \mathrm{T}\overline{\psi}(x)\mathcal{O}^{\mu\nu\rho\dots}_{\alpha\beta\delta\dots}\psi(0)\,| P_i \rangle$$
$$\mathcal{G}^{\mu\nu\rho\dots}_{\alpha\beta\delta\dots} = \sum_j H_j(P,k,\Delta)\,(\tau_j)^{\mu\nu\rho\dots}_{\alpha\beta\delta\dots}(P,k,\Delta)$$



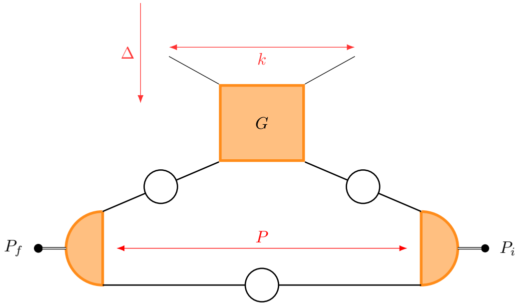
- These interactions probe the partonic distributions and interactions of Hadrons
- **Main idea:** Processes = Hard scattering × Hadronic structure functions
  (Review: Section 2 of Belitsky, Radyushkin, 2005)





Writing the hadronic correlation

- We write the hadronic correlation using elements calculated via functional methods:



- G is the four-point quark correlation function, calculated with scattering equation.
- The quark propagator is calculated via quark DSE.
- The BSWF is calculated via the meson BSE.

(Mezrag, PhD Thesis); (Mezrag, arXiv:1507.05824); (Mezrag et al., 2015)

$$\mathcal{G}^{[\Gamma]}(P,k,\Delta) = \frac{1}{2}\mathrm{Tr}\left[\int dk^- \int \frac{d^4z}{2\pi^4} e^{ik\cdot z} \langle P_f | \overline{\psi}(z)\mathcal{W}\Gamma\psi(0) | P_i \rangle\right]$$

- Partonic distributions are calcuated by integrating the correlator in k^- and taking appropriate traces.

Conclusions

- We explored a new way to calculate the LFWFs and PDAs.
- Very good agreement with the established Nakanishi method.
- We can also tackle extensions to the scalar toy model: unequal masses and complex conjugate mass poles in the propagators — features of many QCD calculations.
- We can calculate beyond the $M^2 = 4m^2$ threshold.
- The contour deformation method can also be applied to other correlation functions — just need a region free of singularities where the path can be deformed.
- We have also explored a way forward for the calculation of partonic structure functions.

Thanks for your attention!

BACKUP

Bethe-Salpeter Wavefunction

- The Bethe-Salpeter Wavefunction (BSWF) appears as the residue of a correlation function $G(p)$:

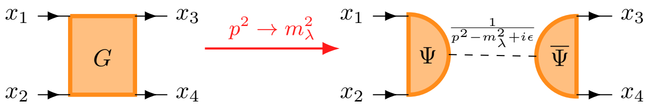
$$\Psi(x,P) = \langle 0|\,\mathrm{T}\phi(0)\phi(x)\,|P\rangle$$
$$\Psi(k,P) = \int d^4x e^{-ik\cdot x}\Psi(x,P)$$

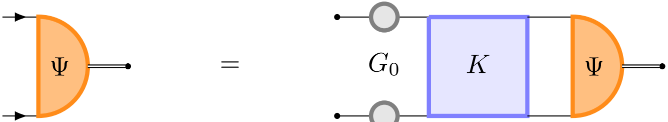
- Determined by the Bethe-Salpeter Equation:

$$\Psi = \mathbf{G_0 K}\Psi$$

$\mathbf{G_0}$ product of the dressed propagators;

$\mathbf{K}$ interaction kernel between the two particles.



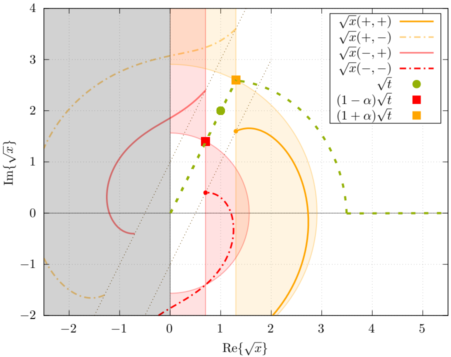


Analytic Structure

$$\mathbf{G_0} = \frac{1}{q_1^2 + m^2}\frac{1}{q_2^2 + m^2}$$

- Poles when $q_{1/2}^2 = -m^2$
- Integration in $\omega \implies$ branch cuts in complex x plane:

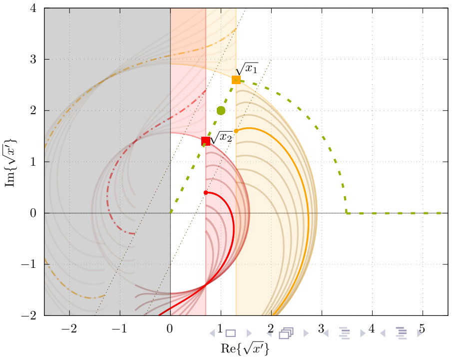
$$\sqrt{x}_{\pm}^{\lambda} = \mp(1 \pm \alpha)\sqrt{t}\left[\omega + i\lambda\sqrt{1 - \omega^2 + \frac{1}{(1 \pm \alpha)^2 t}}\right]$$



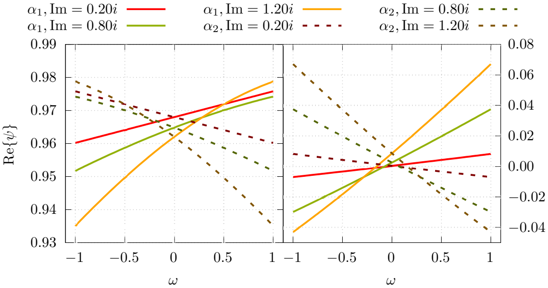
$$\mathbf{K} = \frac{g^2}{(q - q')^2 + \mu^2}$$

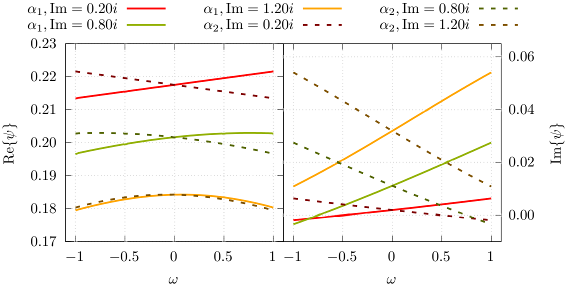
- Poles when $(q - q')^2 = -\mu^2$.
- Branch cuts in complex x' plane:

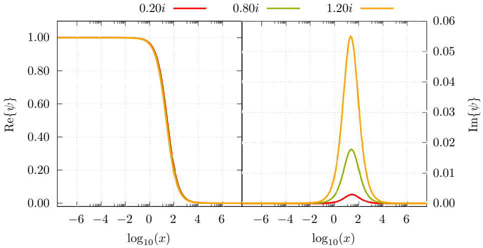
$$\sqrt{x'} = \sqrt{x}\left(\Omega \pm i\sqrt{1 - \Omega^2 + \frac{\beta^2}{x}}\right)$$



BSWF Results







- Small ω and α dependance.
- Symmetry for the combined transformation $\alpha \to -\alpha$ and $\omega \to -\omega$.
- Approximately a monopole:

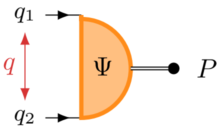
$$\psi \approx \frac{1}{q^2 + \gamma}$$

- **Light-Front Wavefunction** (LFWF): Fourier transform of the BSWF at $x = \lambda n + \vec{x}_\perp$, with n along the light front.

$$\Psi_{LF}(q^+, \vec{q}_\perp, P) = \\ = \mathcal{N} \int dq^- \Psi\left(q^-, q^+, \vec{q}_\perp, P\right),$$

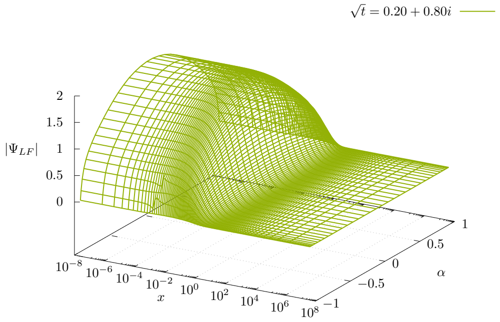
- **Parton Distribution Amplitude** (PDA): Integration of the Ψ_{LF} over $q_\perp$.

$$\phi(\alpha, P) = \int d^2 q_\perp \Psi_{LF}\left(\frac{\alpha}{2}P^+, \vec{q}_\perp, P\right).$$

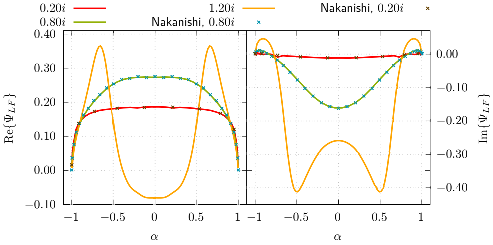


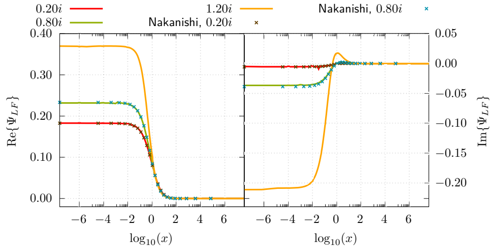
$\xi = \frac{q_1^+}{P^+} = 2\alpha - 1$ is the longitudinal momentum fraction

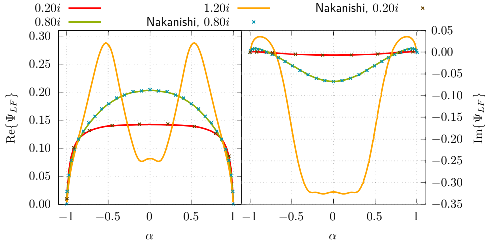
$q = k + \frac{\alpha}{2} P$ is the relative momentum (with $k^+ = 0$).

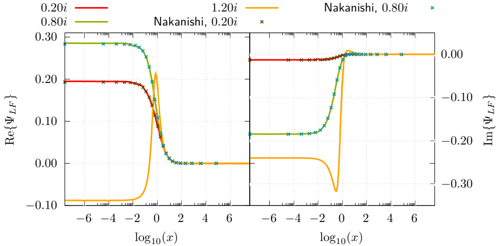


LFWF: α and x dependance



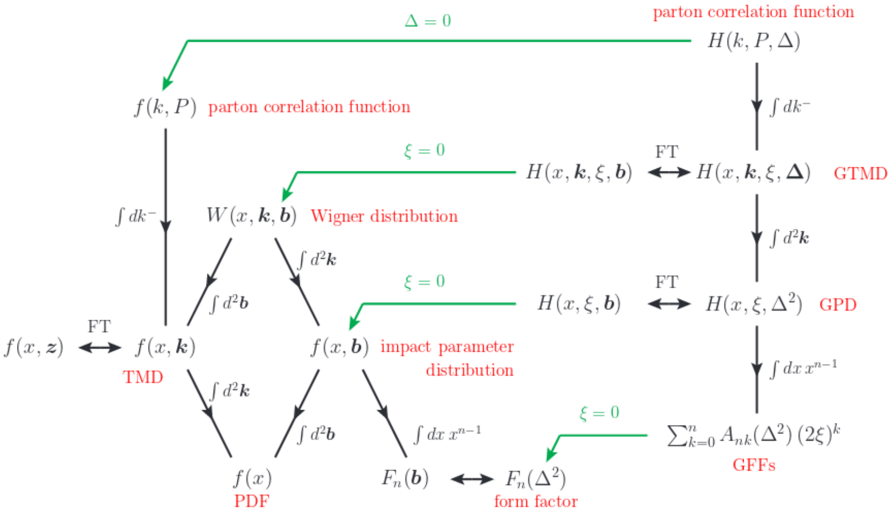






- Symmetric in α.
- Vanishes at $\alpha = \pm 1$.

Moving further



(Picture: Diehl, 2016); (Diehl, 2003); (Meissner, Goeke, Metz, Schlegel; 2008); (Meissner, Metz, Schlegel; 2009)

Unequal masses

- Consider two ϕ of different masses:

$$m_1 = m(1+\varepsilon) \qquad m_2 = m(1-\varepsilon)$$

$$\frac{m_1}{m_2} = \frac{1+\varepsilon}{1-\varepsilon} \qquad 2m = m_1 + m_2$$

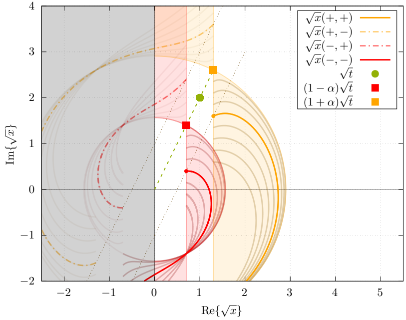
- $\varepsilon \in [-1, 1]$ sets the ratio of the masses
- $\mathbf{G_0}$ is now:

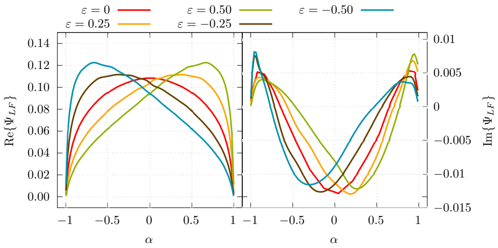
$$\mathbf{G_0} = \frac{1}{q_1^2 + m_1^2} \frac{1}{q_2^2 + m_2^2}$$

- Cuts in x:

$$\sqrt{x}^{\lambda}_{\pm} = \mp(1 \pm \alpha)\sqrt{t} \times \left[\omega + i\lambda\sqrt{1 - \omega^2 + \frac{1}{t}\left(\frac{1 \pm \varepsilon}{1 \pm \alpha}\right)^2}\right]$$

- Integration path still works
- ε adds skewness





Complex Conjugate Masses

- Also consider complex conjugate mass poles:

$$D_\phi(q,m) = \frac{1}{2}\left(\frac{1}{q^2+m^2} + \frac{1}{q^2+(m^*)^2}\right)$$

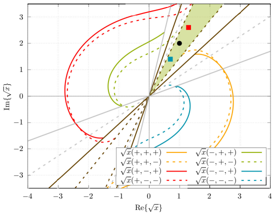
- $m^2 \to m^2(1+i\delta)$, with $m^2, \delta \in \mathbb{R}_+$
- $\mathbf{G_0}$ becomes:

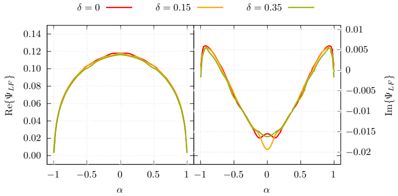
$$\mathbf{G_0} = D_\phi(q_1,m)D_\phi(q_2,m)$$

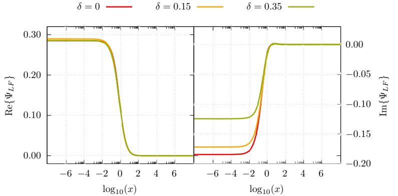
- There are now 8 cuts:

$$\sqrt{x}_{\pm}^{\{\lambda,\nu\}} = \mp(1\pm\alpha)\sqrt{t} \times \left[\omega + i\lambda\sqrt{1-\omega^2+\frac{1}{t}\frac{1+\nu i\delta}{(1\pm\alpha)^2}}\right]$$

- For $\delta < \delta_{crit}$, contour deformation always possible.







Nakanishi Method

- BSWF defined from a smooth weight function $g(x,\alpha)$.

$$\Psi(q,P) = \frac{1}{m^4}\int_0^\infty dx' \int_{-1}^{1} d\alpha' \frac{g(x',\alpha')}{[\kappa + 1 + x' + (1-\alpha'^2)t]^3}, \qquad \kappa = \frac{1}{m^2}\left(q - \frac{\alpha'}{2}P\right)^2.$$

- Light front quantities obtained from the weight function g, for example LFWF:

$$\Psi_{LF} = \frac{\mathcal{N}}{m^2}\int_0^\infty dx' \frac{g(x',\alpha)}{[x' + 1 + x + (1-\alpha^2)t]^2}$$

- The BSE can be rewritten for g:

$$\int_0^\infty dx' \frac{g(x',\alpha)}{[x' + 1 + x + (1-\alpha^2)t]^2} = c\int_0^\infty dx' \int_{-1}^{1} d\alpha' V(x,x',\alpha,\alpha') g(x',\alpha')$$

$$V(x,x',\alpha,\alpha') = \frac{K(x,x',\alpha,\alpha') + K(x,x',-\alpha,-\alpha')}{2\,[x + 1 + (1-\alpha^2)t]}$$

$$K(x,x',\alpha,\alpha') = \int_0^1 dv \frac{\theta(\alpha-\alpha')(1-\alpha)^2}{[v(1-\alpha)(x' + 1 + (1-\alpha'^2)t) + (1-v)C]^2}$$

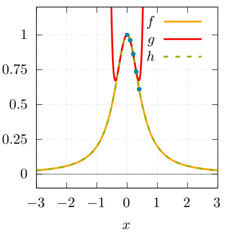
$$C = (1-\alpha')(1 + x + (1-\alpha^2)t) + (1-\alpha)\left(\frac{\beta}{v} + x'\right)$$

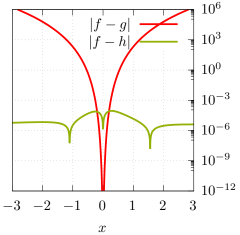
Schlessinger Point Method

- Numerical analytic continuation method:

$$R(\omega) = \cfrac{f(\omega_1)}{1 + \cfrac{a_1(\omega - \omega_1)}{1 + \cfrac{a_2(\omega - \omega_2)}{1 + \cfrac{a_3(\omega - \omega_3)}{\dots}}}}$$

- $\{a_i\}$ obtained by imposing $R(\omega_i) = f(\omega_i)$





- Recurrence relations:

$$R(\omega) = \frac{f(\omega_1)}{1 + \mathcal{Z}_1} = \cfrac{f(\omega_1)}{1 + \cfrac{a_1(\omega - \omega_1)}{1 + \mathcal{Z}_2}} = \dots$$

$$\mathcal{Z}_k = \frac{a_k(\omega - \omega_k)}{1 + \mathcal{Z}_{k+1}} \Leftrightarrow \mathcal{Z}_{k+1} = \frac{a_k(\omega - \omega_k)}{\mathcal{Z}_k} - 1,$$

$$\omega = \omega_k \implies \mathcal{Z}_k = 0$$

$$f(\omega_2) = \frac{f(\omega_1)}{1 + a_1(\omega_2 - \omega_1)},$$

$$\mathcal{Z}_1 = \frac{f(\omega_1)}{f(\omega_2)} - 1 \Leftrightarrow a_1 = \frac{\mathcal{Z}_1}{\omega_2 - \omega_1},$$

Why not do one more iteration?

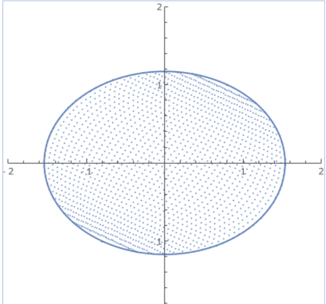
- Do *one more iteration* for a value of $\omega = W \in \mathbb{C}$, with the obtained Ψ

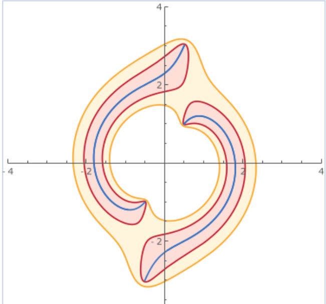
$$\Psi(x, W, t, \alpha) = \mathcal{N} \int_0^\infty dx' \int_{-1}^{1} d\omega' \mathcal{K}(x, x', W, \omega') \Psi(x', \omega', t, \alpha)$$

- **Problem:** Kernel cuts will change
  - For $\omega \in \mathbb{C}$, and $y, \omega' \in [-1, 1]$, Ω turns into a region bounded by the $r(\theta)$ ellipse, with $\omega = a + ib$ and $\sqrt{1 - \omega^2} = c + id$:

$$r(\theta) = \sqrt{a^2 + c^2}\sqrt{\cos^2\theta + E^2 \sin^2\theta} \qquad E = \begin{cases} \frac{d^2}{a^2} & \alpha \neq 0 \\ \frac{b^2}{1+b^2} & \alpha = 0 \end{cases}$$

  - Kernel cuts will eventually overlap





Cuts for complex conjugate mass poles

$$\mathrm{Im}\{\sqrt{\tau}\}\,\mathrm{Re}\{i\sqrt{1+i\delta}\} < \mathrm{Im}\{i\sqrt{1+i\delta}\}\,\mathrm{Re}\{\sqrt{\tau}\}.$$